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RAY THEORY AND SHOCK FORMATION IN RELATIVISTIC ELASTIC SOLIDS

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A theory of rays, or bicharacteristics, is presented within the relativistic framework for the matter scheme of anisotropic elasticity and its generalization to perfectly conducting magnetic bodies. The equation governing the evolution of the amplitude (growth or decay) of weak waves is obtained. This equation allows one to discuss the influence of purely relativistic effects and the effect of initial states germane to the physical description of interest, e.g. high hydrostatic pressure and intense magnetic fields of various settings, on the formation of caustics via the phenomenon of focusing, the resulting steepening of the wave front and the subsequent formation of shock waves. Analytic expressions are obtained for the values of the characteristic parameter corresponding to the breakdown of the weak-wave solution for plausibly simple elastic behaviours and various settings of the initial magnetic field. The relative influence, at both relativistic and non-relativistic orders, of the nonlinearity of the material, the initial pressure, and the direction of the magnetic field with respect to the wave-propagation direction and the elastic-disturbance polarization is thoroughly discussed, some effects favouring, others delaying, the formation of shocks. For magnetic bodies the present treatment simultaneously provides results that prove to be useful in the non-relativistic theory of magnetoelasticity in perfect conductors.

1. Introduction

In a relatively recent paper (Maugin 1977), by using a simplified model of relativistic *isotropic* elasticity and the distribution-theory method already used by Lichnerowicz (1967, 1971) in relativistic hydrodynamics and magnetohydrodynamics, it has been qualitatively proven that so-called weak, or infinitesimal, discontinuities propagating throughout a prestressed elastic body could form shocks. That is, at one point along the bicharacteristics associated with the wave-equation system, the infinitesimal-discontinuity solution might become unbounded for certain initial states and types of wave fronts, so that a strong discontinuity or shock solution might be envisaged from then on. The qualitative nature of the proof obviously hindered the finding of the critical value of the time-like parameter for which this strengthening of the wave solution occurs. The quasilinear hyperbolic structure of the system of equations for the *exact* theory of *nonlinear isotropic* relativistic elasticity, as developed in Maugin (1978*d*), is more or less obvious (cf. Hughes *et al.* 1977), so that the same type of phenomenon is also to be expected in this exact theory (see general treatises on the matter, e.g. Whitham (1974), Jeffrey (1976)). The embryonic study of shock waves for one-dimensional motions that has been given (Maugin, 1978*f*, 1979*a*) is therefore justified.

The primary purpose of the present paper is to provide a *quantitative* proof of the strengthening of wave fronts in relativistic elasticity, in a more general framework that might better correspond to the physical situations encountered in certain dense astrophysical objects such as neutron stars. Indeed, the external crust of such bodies, although built up of a highly incompressible but elastic solid, does not behave isotropically, and probably behaves more like a cubic structure (cf. Lamb 1977). Since in addition no linearization must be performed beforehand because of the possibly extremely high pressures involved, we need an *exact* (i.e. with finite deformations) relativistic theory of *elastically anisotropic* bodies. Such a theory was deduced in an early work (Maugin 1971) from a variational principle in a general relativistic framework. Moreover, the quantitative nature of the present work forces us to consider, in place of Lichnerowicz's qualitative method, the quantitative and geometrical description of singular surfaces given for relativistic continuum mechanics in Maugin (1976). This description explicitly involves the geometry of the wave front at the second order. Finally, another purpose of the work is to show clearly both the effects of initial states, such as a high hydrostatic pressure, and the purely relativistic effects on the strengthening of the wave fronts. Technical complexities, however, reduce the generality of the work. First, only a propagation through a spatially homogeneous state of strains and stresses ahead of the wave front will be envisaged. A general inhomogeneous state could be considered at the price of lengthier calculations which would not change the nature of the results much. Secondly, simple applications will refer to plane wave fronts and to bodies that behave almost like Hookean bodies. General expressions will nonetheless be obtained before those simplifications. Lastly, because we consider an anisotropic body we shall, in the end, study only *a priori* longitudinally and transversely polarized wave fronts. Note also that the study of wave propagation through a definite initial state would require as a premise the proof of the existence and uniqueness of this initial state. This proof, however, is far beyond our present capabilities for the general relativistic, anisotropic elastic scheme of matter (as it is also for its magnetoelastic generalization), so that the existence and uniqueness of such a state will be assumed, even though some hints at this proof are probably contained in the work of Hughes *et al.* (1977).

In other recently published papers (Maugin 1978 *b*, *c*, *e*) the pertinence has also been emphasized of taking intense magnetic fields into account in the studies concerned with wave-like motions in the above-mentioned physical situations. Does not the crust of dense solid-like stars offer the best example of a perfect conductor of electricity? Although both classical and relativistic magnetohydrodynamic schemes have been the object of numerous works as far as wave motion is concerned (see, for example, Jeffrey & Taniuti (1964) for the classical framework and Lichnerowicz (1971) for the relativistic one), nonlinear wave motions in magnetoelasticity have received much less attention and not until the exhaustive work of Bazer & Ericson (1974) did one have a unique and definitive work on these in the *classical* framework. Nonetheless, elements of nonlinear wave motion in the relativistic framework were developed in Maugin (1978 *e*) with a view to studying some aspects of the behaviour of dense astrophysical magnetic objects where a relativistic treatment may be justified. This was achieved for bodies that behave isotropically in both their mechanical and magnetic properties. In the present paper we shall consider the magnetoelastic case of *elastically anisotropic* perfect conductors of electricity as a generalization of the purely elastic case. Again, the basic theory needed in such a development has been given in Maugin (1971) insofar as constitutive equations are concerned (see also Maugin 1978 *a*).

Section 2 is devoted to reviewing the essentials of relativistic anisotropic elasticity and magnetoelasticity. In § 3 the general wave equation governing infinitesimal-discontinuity modes in the purely thermoelastic case is obtained. A similar development is given in § 4 for the magnetoelastic scheme of matter. There it is proven that Hadamard's hyperbolicity condition is always reinforced

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by an initial magnetic field of any setting so that, if weak-wave fronts can propagate in the purely elastic case, they can propagate in the magnetoelastic case as well. The exact expression for invariant speeds for these weak waves is given for *a priori* polarized wave fronts and simple settings of the initial magnetic field. This limitation to peculiar polarizations of elastic perturbances results from the complex treatment due to the general elastic anisotropy of the body (the difficulty is of the same order as that encountered in studying wave-like motions in crystallographic structures). The ray or bicharacteristic theory is first developed in § 5 for a propagation through a spatially homogeneously stressed initial state of nonlinear elastic matter in the absence of magnetic fields. This sketch of the theory culminates with the obtaining of the general equation governing the evolution of the four-dimensional vector amplitude of infinitesimal discontinuities along the corresponding rays. In this state of generality the local geometry (curvature, tangential derivatives) at the second order of the wave front intervenes in the equation. The equation is then applied in § 6 to a brief discussion of the evolution (growth or decay) of the amplitude of *a priori* polarized *flat* wave fronts. The problem is then specialized by considering an elastic behaviour closely resembling that of a linear isotropic Hookean body. An analytic expression is obtained for the break-down value of the relevant parameter (distance along the wave-front normal); therefrom caustic formation due to focusing (in the language of geometric optics) will force us to consider *shock waves* in place of infinitesimal discontinuities. The critical value is entirely expressed in terms of the initial strength of the wave front, the usual longitudinal elastic-disturbance speed, and two parameters (which are not necessarily small in the relativistic context) that characterize relativistic effects and the effects resulting from an initial state of high hydrostatic pressure, respectively. The accompanying discussion clearly shows that a sufficiently strong nonlinearity in the elastic behaviour is needed to prove that shocks may form through the steepening of compressive infinitesimal discontinuities. Relativistic effects, however, do not necessarily favour this phenomenon.

In § 7 the ray theory developed in previous sections for the purely elastic case is generalized to the magneto-elastic case and the evolution equation for the four-vector amplitude of weak waves along rays of the wave system is obtained for a spatially homogeneously stressed and *magnetized* state ahead of the wave front. The equation thus obtained involves the geometry of the wave front and relativistic corrections which themselves involve both the initial stresses and the initial magnetic field as well as the nonlinearity of the elastic body (via third-order elastic coefficients). To shed light on the phenomenon of growth or decay of the wave-front amplitude *plane* wave fronts propagating through a spatially homogeneous state of high hydrostatic pressure and intense magnetic field are carefully examined in § 8 when the elastic constitutive equation assumes a simple plausible form. The study reveals the following behaviour of the weak-wave amplitude and influence of the magnetic field. For elastically longitudinally polarized wave fronts (all wave fronts are transverse from the electromagnetic viewpoint), compressive wave fronts can eventually form shocks if the nonlinear elastic behaviour presents a marked concavity in the stress–strain relation, but a transverse magnetic field makes this requirement less stringent compared with the purely elastic case. Other magnetic effects appear at the relativistic order and may or may not favour the shock-formation phenomenon depending on the orientation of the initial magnetic field with respect to the propagation direction. For elastically transversely polarized wave fronts it is shown that a longitudinal magnetic field may delay the formation of a transverse magneto-elastic shock in a nonlinear elastic body in a state of high hydrostatic pressure, and this at the non-relativistic order, whereas a purely transverse magnetic field may have the converse effect, but at the relativistic order. These conclusions might encourage the study of magnetoelastic shock waves on a relativistic basis. All necessary ingredients for such a study, including the related Hugoniot relation, are contained in Maugin (1978*c*).†

A final remark is in order about the proposed relativistic treatment. Obviously, this treatment contains in the so-called non-relativistic limit the classical, non-relativistic, results. This is particularly advantageous for the magneto-elastic scheme for which few results seem to be known in the classical theory of magnetoelasticity. As for magnetohydrodynamics (cf. Lichnerowicz 1971; Jeffrey & Taniuti 1964), it appears here also that a relativistic treatment, by unifying the time and space concepts and having recourse to space–time geometrical objects, is algebraically simpler than a classical treatment. In this respect the contents of the present paper compare favourably with those of papers by McCarthy (1968, 1969) devoted to the growth of classical magnetoelastic waves in isotropic bodies.

2. Preliminaries

We consider the same notation as in Maugin (1978*a–e*). Space–time $M = (V^4, \boldsymbol{g})$ is a differentiable manifold of dimension four, of continuity class $C^p, p > 2$, equipped with a normal hyperbolic metric $\boldsymbol{g}$, and hence with Lorentzian signature $+2$. A local chart in M is given by $\{x^\alpha$, $\alpha = 1, 2, 3, 4$; index 4 time-like$\}$. For notational convenience and unless otherwise specified the velocity of light in vacuum is set equal to one: $c = 1$. The parameter u^α is the four-velocity such that $g_{\alpha\beta}u^\alpha u^\beta + 1 = 0$. The partial and covariant derivatives with respect to $\{x^\alpha\}$ in M are denoted by ∂_α and ∇_α. The gradient operator in the direction of the four-vector field $\boldsymbol{A}$ is denoted by $\mathrm{D}_A = \boldsymbol{A}\cdot\boldsymbol{\nabla}$. The spatial projector $\boldsymbol{P} = \{P^{\beta}_{.\alpha} = \delta^\beta_\alpha + u^\beta u_\alpha\}$ is used systematically in the following development to write down the local canonical space–time decomposition of any tensor field $\boldsymbol{T}$ defined on M. The local spatial projection of any geometrical object $\boldsymbol{T}$ is given by $(\boldsymbol{T})_\perp$ and admits $\boldsymbol{u}$ as zero vector for all its indices in a local chart. Objects such as $\boldsymbol{T} = (\boldsymbol{T})_\perp$ are said to be *spatial*. The transverse or spatial covariant derivative is defined by $\boldsymbol{\nabla}^\perp = (\boldsymbol{\nabla})_\perp$.

Let $\mathscr{M} = (V^3, \boldsymbol{G})$ be the three-dimensional manifold that serves to describe the *material* continuum. $\mathscr{M}$ is equipped with the local background metric $\boldsymbol{G}$ which is said to be *invariant* in the sense that $\mathrm{D}_u \boldsymbol{G} = \boldsymbol{0}$ always. Tensor fields defined on $\mathscr{M}$ are said to be *material*. The material body B, whose constituents are the material 'particles' $\boldsymbol{X}$ of proper time τ, is an open, regular, simply connected subset of $\mathscr{M}$. A local chart in $\mathscr{M}$ is given by $\{X^K, K = 1, 2, 3\}$. As in Maugin (1971) and Carter & Quintana (1972), we admit that the relativistic motion of the material body B is described by means of a *canonical differentiable projection* $\mathscr{P}$ such that $\mathscr{P}\colon \mathscr{B} = \mathscr{T}[B] \to \mathscr{M}$. Here $\mathscr{B}$ is the open tube of M that is swept out by B. We have thus

$$\mathscr{P}\colon X^K = \tilde{X}^K(x^\alpha), \quad \tau = \tilde{\tau}(x^\alpha), \quad \boldsymbol{x} \in \mathscr{C}_X \subset \mathscr{B}, \tag{2.1}$$

where $\mathscr{C}_X$ is the time-like world line of $\boldsymbol{X}$, a curve parametrized by τ at fixed $\boldsymbol{X}$. If $\mathscr{P}$ is $C^1(\mathscr{B})$, then

$$X^K_\alpha = \partial_\alpha \tilde{X}^K, \quad X^K_\alpha u^\alpha = \mathrm{D}_u \tilde{X}^K = 0 \tag{2.2}$$

defines the so-called *inverse motion gradient*. Central to the notion of finite strain for an *anisotropic medium* in a relativistic background is the material tensor $\mathscr{C}$ defined as being the image of the

† Preliminary results concerning the contents of §§3 and 5 were given in short notes (Maugin 1979*b*, *c*). Heat-conducting thermoelastic materials were recently considered by Ukeje (1980).

reciprocal space–time metric, $\boldsymbol{g}^{-1}$, by the projection $\mathscr{P}$ of space–time M on its quotient by the congruence of world lines $\mathscr{C}_X(\tau)$, $\boldsymbol{X} \in B$, (Maugin 1971). That is†,

$$\mathscr{C} = \mathscr{P}(\boldsymbol{g}^{-1}) = \mathscr{P}(\boldsymbol{P}^{-1}) \tag{2.3}$$

or, in component form,

$$\mathscr{C}^{KL} = g^{\alpha\beta} X^K_\alpha X^L_\beta = P^{\alpha\beta} X^K_\alpha X^L_\beta = \mathscr{C}^{LK}. \tag{2.4}$$

This material tensor is the relativistic analogue of the Piola finite-strain tensor of classical non-linear continuum mechanics (cf. Eringen 1967, ch. I).

In the absence of dissipative processes, spin effects and electromagnetic fields, after Legendre transformation of the energy density, we extract from Maugin (1971) the following set of field equations for the exact finite-strain, anisotropic, adiabatic, general relativistic theory of elasticity at any regular point in $\mathscr{B}$:

$$A^{\alpha\beta} = kT^{\alpha\beta} \quad \text{(Einstein equations)}, \tag{2.5}$$

$$\nabla_\alpha(\rho u^\alpha) = 0 \quad \text{(continuity equation)}, \tag{2.6}$$

$$\nabla_\alpha(\rho\eta u^\alpha) = 0 \quad \text{(entropy balance)}, \tag{2.7}$$

$$\nabla_\beta T^{\alpha\beta} = 0 \quad \text{(balance of energy–momentum)}, \tag{2.8}$$

where ρ is the proper density of matter, η is the proper density of entropy, $A^{\alpha\beta}$ is the Einstein tensor, k is the gravitational constant in *ad hoc* units and $T^{\alpha\beta}$ is the total symmetric energy–momentum tensor. The whole physical description of matter is contained in $T^{\alpha\beta}$. The following hypotheses are considered:

(H 1) The body B is *nonlinear elastic*.

(H 2) It has a *general anisotropic structure* insofar as its mechanical properties are concerned.

Note that (H 2) is less stringent than the corresponding hypothesis considered in Maugin (1978 *d*), where the body was supposed to behave isotropically in its mechanical properties. The relaxation of this hypothesis will complicate the present treatment while, all other things being kept unchanged, allowing us better to describe physical reality. With $\boldsymbol{T}$ admitting the following simple canonical space–time decomposition:

$$T^{\alpha\beta} = \rho(1+\epsilon)\, u^\alpha u^\beta - t^{\alpha\beta}, \tag{2.9}$$

where ϵ is the internal energy per unit of proper mass and $t^{\alpha\beta}$ is the spatial symmetric stress tensor, the constitutive equations needed are deduced on a thermodynamical basis from ϵ in accordance with the equations (see, for example, Maugin 1978 *a*)

$$\boldsymbol{t} = -2\rho\mathscr{P}\,\partial\tilde{\epsilon}/\partial\mathscr{C} \quad \text{or} \quad t_{\alpha\beta} = -2\rho(\partial\tilde{\epsilon}/\partial\mathscr{C}^{KL})\, X^K_\alpha X^L_\beta, \tag{2.10}$$

and

$$\theta = \partial\tilde{\epsilon}/\partial\eta > 0 \quad (\inf\theta = 0), \tag{2.11}$$

with

$$\epsilon = \tilde{\epsilon}(\mathscr{C}, \eta). \tag{2.12}$$

The function $\tilde{\epsilon}$ is assumed to be at least C^3 on its domain of definition $D = \mathbb{R}^6 \times \mathbb{R}^+$ (since $\mathscr{C} = \mathscr{C}^{\mathrm{T}}$, $\eta > 0$; $T =$ transpose). The parameter θ is none other than the proper thermodynamical temperature. By using the notation introduced above, and projecting (2.8) along $\boldsymbol{u}$ and orthogonally to it, we can rewrite (2.6)–(2.8) in the following enlightening form

$$\mathscr{C} \equiv \mathrm{D}_u\rho + \rho\nabla_\alpha u^\alpha = 0 \quad \text{(continuity)}, \tag{2.13}$$

$$\mathscr{N} \equiv \rho\theta\mathrm{D}_u\eta = 0 \quad \text{(isentropic evolution)}, \tag{2.14}$$

$$\mathfrak{M}^\alpha \equiv \rho f^{\alpha}_{\cdot\beta}\mathrm{D}_u u^\beta - P^{\alpha}_{\cdot\gamma}\nabla^{\perp}_\beta t^{\beta\gamma} = 0 \quad \text{(Euler–Cauchy motion equations)}, \tag{2.15}$$

and

$$\mathscr{E} \equiv \rho\,\mathrm{D}_u\epsilon - t^{\alpha\beta}\nabla_\alpha u_\beta = 0 \quad \text{(energy equation)}, \tag{2.16}$$

† The notation $\mathscr{C}$ replaces the inconvenient $\boldsymbol{C}^{-1}$, used in previous work.

where the spatial symmetric tensor defined by

$$f^{\alpha\beta} = (1+\epsilon)\, P^{\alpha\beta} - \rho^{-1} t^{\alpha\beta} \tag{2.17}$$

is called the tensorial index of the material (Maugin 1977). This last notion generalizes the notion of thermodynamical index introduced by Lichnerowicz (1971) for relativistic hydrodynamics. The scalar index can be defined by (tr ≡ trace)

$$f = \tfrac{1}{3}\,\mathrm{tr}\,\boldsymbol{f}. \tag{2.18}$$

In particular, for a state of hydrostatic pressure p_0, $t^{\alpha\beta} = -p_0 P^{\alpha\beta}$, and (2.18) yields Lichnerowicz's index $f_0 = 1+\epsilon+p_0/\rho$.

If we now account for the presence of electromagnetic fields and envisage a *magnetoelastic* scheme for the matter under consideration, then in addition to (H 1), (H 2) we consider the hypotheses:

(H 3) In spite of (H 2) and to simplify the following development to some extent, the body behaves *isotropically* in its magnetic properties.

(H 4) Coupled effects such as pyromagnetism, piezomagnetism and magnetostriction are ignored.

(H 5) The magnetic permeability μ (a constant) of B is not very different from one.

(H 6) B is a perfect conductor of electricity.

On combining the above equations with the results of Maugin (1978*a*), on account of hypotheses (H 3)–(H 6) it is shown that an additional term

$$T_{\mathrm{M}}^{\alpha\beta} = \mu[\mathscr{H}^2(\tfrac{1}{2}g^{\alpha\beta} + u^\alpha u^\beta) - \mathscr{H}^\alpha \mathscr{H}^\beta] = T_{\mathrm{M}}^{\beta\alpha} \tag{2.19}$$

must be added in the right-hand side of (2.9). $\mathscr{H}$ is the (spatial) magnetic-field four-vector. The whole set of Maxwell's equations is contained in the covariant equation (Lichnerowicz 1971)

$$\nabla_\alpha(u^\alpha \mathscr{H}^\beta - u^\beta \mathscr{H}^\alpha) = 0. \tag{2.20}$$

While equations (2.13) and (2.14) remain unchanged, on account of the additional contribution (2.19) equations (2.15) and (2.16) transform to

$$\tilde{\mathfrak{M}}^\alpha = \rho \tilde{f}^{\alpha}_{\cdot\beta} \mathrm{D}_u u^\beta - P^{\alpha}_{\cdot\gamma} \nabla^{\perp}_{\beta} t^{\beta\gamma} - \mu[\mathscr{H}^\alpha \nabla_\beta \mathscr{H}^\beta + \mathscr{H}^\beta (\nabla_\beta \mathscr{H}^\alpha)_\perp - \nabla^{\perp\alpha}(\tfrac{1}{2}\mathscr{H}^2)] = 0 \tag{2.21}$$

and

$$\tilde{\mathscr{E}} = \rho\, \mathrm{D}_u[\epsilon + (\tfrac{1}{2}\mu \mathscr{H}^2/\rho)] - (t^{\alpha\beta} + t_{\mathrm{M}}^{\alpha\beta})\, \nabla^{\perp}_{(\beta} u_{\alpha)} = 0, \tag{2.22}$$

where

$$t_{\mathrm{M}}^{\alpha\beta} = \mu(\mathscr{H}^\alpha \mathscr{H}^\beta - \tfrac{1}{2}\mathscr{H}^2 P^{\alpha\beta}) = t_{\mathrm{M}}^{\beta\alpha} \tag{2.23}$$

and

$$\tilde{f}^{\alpha\beta} = f^{\alpha\beta} + (\mu \mathscr{H}^2/\rho)\, P^{\alpha\beta} = \tilde{f}^{\beta\alpha} (\tilde{f}^{\alpha\beta} u_\beta = 0). \tag{2.24}$$

The former spatial symmetric tensor t_{M} is none other than the spatial (Maxwell) magnetic stress tensor in perfect conductors. The space–time decomposition of (2.20) yields

$$u^\alpha u^\beta \nabla_\alpha \mathscr{H}_\beta + \nabla_\alpha \mathscr{H}^\alpha = 0 \tag{2.25}$$

and

$$\mathscr{H}^\beta(\nabla_\alpha u^\alpha) + (\mathrm{D}_u \mathscr{H}^\beta)_\perp - \mathscr{H}^\alpha(\nabla_\alpha u^\beta)_\perp = 0. \tag{2.26}$$

The following equations are direct consequences of (2.25), (2.26) and (2.21), (2.22) (Maugin 1978*e*):

$$\mathscr{H}^2(\nabla_\alpha u^\alpha) + \mathrm{D}_u(\tfrac{1}{2}\mathscr{H}^2) + u^\beta \mathscr{H}^\alpha \nabla^{\perp}_{\alpha} \mathscr{H}_\beta = 0 \tag{2.27}$$

and

$$\rho(1+\epsilon)\, \nabla_\beta \mathscr{H}^\beta + \mathscr{H}_\alpha \nabla_\beta t^{\alpha\beta} = 0. \tag{2.28}$$

For subsequent use we also note the identities

$$X^K_\alpha u^\alpha = 0, \quad \mathscr{H}_\alpha u^\alpha = 0. \tag{2.29a, b}$$

We now have at hand all the equations needed in the study of nonlinear wave propagation in elastically anisotropic, magnetic, perfect conductors in the general relativistic framework.

3. The equation in the purely elastic case

Let us denote by

$$S = \{\boldsymbol{g}, \boldsymbol{u}, X^K_\alpha, \rho, \eta, t^{\alpha\beta}, \theta\}, \tag{3.1}$$

if it exists and is unique, the state or solution at a regular point $\boldsymbol{x} \in \mathscr{B}$ of the system of equations given in § 2. A wave front of equation

$$W(x^\alpha) = 0 \tag{3.2}$$

or, in a Gaussian representation,

$$x^\alpha = \phi^\alpha(a^\Gamma;\ \Gamma = 1, 2, 3), \tag{3.3}$$

that propagates through $\mathscr{B}$ and separates at each instant this open tube into two open regions $\mathscr{B}^+$ (ahead of W) and $\mathscr{B}^-$ (behind W), such that $\mathscr{B} = \mathscr{B}^+ + \mathscr{B}^- + W$, is said to be an infinitesimal-discontinuity front, or weak discontinuity, in the present context if, with $\boldsymbol{g} \in C^{p>2}(\mathscr{B})$, $(\boldsymbol{u}, \rho, \eta, X^K_\alpha) \in C^{0,1}(\mathscr{B})$ (i.e. continuously differentiable on $\mathscr{B}$ and with first-order derivatives piecewise continuous on $\mathscr{B}$) the fields $\nabla\boldsymbol{u}$, $\nabla\rho$, $\nabla\eta$, ∇X^K_α, and hence $\nabla\mathscr{C}$, suffer discontinuity jumps across W. Using the formalism of Maugin (1976) and in agreement with Hadamard's lemma on singular surfaces (W is a singular surface of order one in Hadamard's classification), we shall denote by $\bar{U}^\alpha$ the fundamental infinitesimal-discontinuity four-vector amplitude such that

$$[\![\nabla_\beta u^\alpha]\!] = N_\beta \bar{U}^\alpha, \quad \bar{U}^\alpha = [\![\mathrm{D}_N u^\alpha]\!], \quad \bar{U}^\alpha u_\alpha = 0, \tag{3.4a, b}$$

where

$$[\![A]\!] = A^+ - A^-, \quad N_\alpha = \partial_\alpha W/(g^{\mu\gamma}\,\partial_\mu W\,\partial_\gamma W)^{\frac{1}{2}}, \tag{3.5}$$

$A^+ \in S^+$ and $A^- \in S^-$ being the uniform limits of the field A (which is discontinuous across W) when approaching W on its two sides respectively and the unit normal $\boldsymbol{N}$ being oriented from the 'minus' side to the 'plus' side. The fact that we assume $g^{\alpha\beta}N_\alpha N_\beta = 1$ implies that $\boldsymbol{N}$ is *a priori* space-like. On account of the assumed differentiability, Einstein's equations (2.5) do not intervene in the wave propagation problem, and gravitational waves are excluded from the treatment. Similarly to (3.4), we set

$$[\![\nabla_\beta X^K_\alpha]\!] = N_\beta \bar{B}^K_\alpha, \quad \bar{B}^K_\alpha = [\![\mathrm{D}_N X^K_\alpha]\!], \tag{3.6}$$

$$[\![\nabla_\beta \rho]\!] = N_\beta \bar{P}, \quad \bar{P} = [\![\mathrm{D}_N \rho]\!], \tag{3.7}$$

and

$$[\![\nabla_\beta \eta]\!] = N_\beta \bar{\eta}, \quad \bar{\eta} = [\![\mathrm{D}_N \eta]\!]. \tag{3.8}$$

In agreement with Maugin (1976) we also introduce the scalar quantities N_0 and $\mathscr{U}$ by

$$N_0 = -u^\alpha N_\alpha, \quad \mathscr{U} = N_0/(1+N_0^2)^{\frac{1}{2}}. \tag{3.9}$$

The latter is none other than the *intrinsic speed* of propagation of W, i.e. its non-dimensional speed as measured relative to the moving matter. We wish to express the quantities $\bar{B}^K_\alpha$ and $\bar{P}$ in terms of the fundamental amplitude $\bar{U}^\alpha$. To that purpose we note from (2.29*a*), (2.13), (3.4) and (3.5) that

$$u^\alpha [\![\nabla_\beta X^K_\alpha]\!] + [\![\nabla_\beta u^\alpha]\!]\, X^K_\alpha = 0. \tag{3.10}$$

Hence

$$\begin{aligned} u^\alpha [\![\nabla_\beta X^K_\alpha]\!] &= -X^K_\alpha \bar{U}^\alpha N_\beta \\ &= u^\alpha [\![\nabla_\alpha X^K_\beta]\!] = -N_0 \bar{B}^K_\beta = -X^K_\sigma \bar{U}^\sigma N_\beta, \end{aligned} \tag{3.11}$$

and
$$[\![\mathrm{D}_u\rho]\!] = u^\alpha[\![\nabla_\alpha\rho]\!] = -\rho[\![\nabla_\gamma u^\gamma]\!] = -N_0\bar{P} = -\rho(N_\gamma \bar{U}^\gamma), \tag{3.12}$$
so that
$$\bar{P} = \rho N_0^{-1}(N_\sigma \bar{U}^\sigma), \quad \bar{B}^K_\alpha = N_0^{-1}X^K_\sigma \bar{U}^\sigma N_\alpha. \tag{3.13}$$
The first of (3.11) holds good because $\boldsymbol{g}\in C^3$. We thus note for further use that
$$[\![\nabla_\beta X^K_\alpha]\!] = N_0^{-1}N_\beta X^K_\sigma \bar{U}^\sigma N_\alpha, \quad [\![\nabla_\beta\rho]\!] = \rho N_0^{-1}(N_\sigma \bar{U}^\sigma)\, N_\beta \tag{3.14}$$
and
$$[\![\mathrm{D}_u X^K_\alpha]\!] = -X^K_\sigma \bar{U}^\sigma N_\alpha, \quad [\![\mathrm{D}_u\rho]\!] = -\rho(N_\alpha \bar{U}^\alpha). \tag{3.15}$$
Finally, the unit *spatial* four-vector in the direction of propagation of W is given by
$$\left.\begin{aligned}\lambda_\alpha = N^\perp_\alpha/(1+N_0^2)^{\frac{1}{2}}, \quad N^\perp_\alpha = (N_\alpha)_\perp,\\ \lambda_\alpha u^\alpha = 0, \quad \lambda_\alpha P^{\alpha\beta}\lambda_\beta = 1.\end{aligned}\right\} \tag{3.16}$$
On taking the jump of (2.14) and (2.15), we obtain
$$\rho\theta N_0\bar{\eta} = 0 \quad \text{or} \quad \rho\theta\mathscr{U}\bar{\eta} = 0 \tag{3.17}$$
and
$$[\![\mathfrak{M}^\alpha]\!] = \rho f^\alpha_{\cdot\beta}[\![\mathrm{D}_u u^\beta]\!] - P^\alpha_{\cdot\gamma}P^{\beta\lambda}[\![\nabla_\lambda t_\beta{}^\gamma]\!] = 0. \tag{3.18}$$
With the help of (2.10), (3.9) and (3.14) it is shown that
$$P^\alpha_{\cdot\beta}P^\lambda_{\cdot\beta}[\![\nabla_\lambda t^{\beta\gamma}]\!] = -\bar{Q}^{\alpha\sigma}(\boldsymbol{\lambda};\, S^+)\,[(1+N_0^2)/N_0]\,\bar{U}_\sigma - \mathscr{B}^{\beta\alpha}(S^+)\,[\![\nabla^\perp_\beta\eta]\!], \tag{3.19}$$
where
$$\bar{Q}^{\alpha\sigma}(\boldsymbol{\lambda};\, S^+) = \bar{C}^{\beta\alpha\mu\sigma}(S^+)\,\lambda_\beta\lambda_\mu, \tag{3.20}$$
$$\bar{C}^{\beta\alpha\mu\sigma}(S^+) = [C^{\beta\alpha\mu\sigma}_{\mathrm{E}} - t^{\beta\alpha}P^{\mu\sigma} - t^{\sigma\alpha}P^{\beta\mu} - t^{\beta\sigma}P^{\alpha\mu}]\,(S^+), \tag{3.21}$$
$$C^{\beta\alpha\mu\sigma}_{\mathrm{E}}(S^+) = 4\left(\rho\frac{\partial^2\tilde{\epsilon}}{\partial\mathscr{C}^{KL}\,\partial\mathscr{C}^{MN}}X^{K\beta}X^{L\alpha}X^{M\mu}X^{N\sigma}\right)(S^+) = C^{(\beta\alpha)(\mu\sigma)}_{\mathrm{E}} = C^{\mu\sigma\beta\alpha}_{\mathrm{E}}, \tag{3.22}$$
and
$$\mathscr{B}^{\beta\alpha}(S^+) = 2\left(\rho\frac{\partial^2\tilde{\epsilon}}{\partial\eta\,\partial\mathscr{C}^{KL}}X^{K\beta}X^{L\alpha}\right)(S^+), \tag{3.23}$$
with $X^{K\beta} = P^{\beta\alpha}X^K_\alpha$. We shall say that a linear operator of the set of symmetric second-order tensors onto itself is *Hookean* if and only if it is symmetric. Therefore $\boldsymbol{C}_{\mathrm{E}}$ is Hookean while $\bar{\boldsymbol{C}}$ is not. The former is the spatial tensor of second-order adiabatic elasticities with, at most, 21 independent components in a local chart $\{x^\alpha\}$. These components are evaluated at S^+. The spatial tensor $\bar{\boldsymbol{C}}$, which accounts for the state $t^{\alpha\beta}(S^+)$ of stresses ahead of the wave front, may be called the tensor of *apparent* (or effective) second-order elasticities at S^+. The symmetric spatial tensor $\mathscr{B}^{\alpha\beta}$ is the tensor of thermoelastic coefficients. $\bar{\boldsymbol{Q}}$ is the spatial Christoffel *acoustic tensor* of the purely thermoelastic case. It is clearly symmetric since in intrinsic notation
$$\bar{\boldsymbol{Q}} = \boldsymbol{\lambda}\boldsymbol{C}_{\mathrm{E}}\boldsymbol{\lambda} - \{[(\boldsymbol{\lambda}\cdot\boldsymbol{t})\otimes\boldsymbol{\lambda}] + [(\boldsymbol{\lambda}\cdot\boldsymbol{t})\otimes\boldsymbol{\lambda}]^{\mathrm{T}} + \boldsymbol{t}\}. \tag{3.24}$$
It depends on both the state ahead of W and the direction of propagation.

Equation (3.17) imposes with N_0 or $\mathscr{U} \neq 0$ that $\bar{\eta} = 0$ across the wave front W which, if it is to propagate at all ($\mathscr{U} \neq 0$), is an *isentropic wave front* if and only if there is a non-zero transfer of mass and energy across it. The last contribution in (3.19) can thus be discarded by virtue of (3.17).

Upon substituting the expression (3.19) into (3.18) and taking the definition of $\mathscr{U}$ into account, we obtain the wave equation in the form

$$\bar{H}^{\alpha\sigma}(\mathscr{U}^2, \lambda; S^+)\,\bar{U}_\sigma = 0, \tag{3.25}$$

where $$\bar{H}^{\alpha\sigma} = \rho^+ f^{\alpha\sigma}(S^+)\,\mathscr{U}^2 - \bar{Q}^{\alpha\sigma}(S^+) = \bar{H}^{\sigma\alpha}, \quad \bar{H}^{\alpha\sigma}u_\alpha = 0. \tag{3.26}$$

The compatibility condition for solving (3.25) for $\bar{U}_\sigma$ in general reads $\det|\bar{H}| = 0$, which provides a cubic in $\mathscr{U}^2$. A *strong ellipticity* condition of the form

$$c_\alpha \bar{Q}^{\alpha\sigma} c_\sigma > 0 \tag{3.27}$$

for any spatial unit four-vector c will ensure the hyperbolicity in Hadamard's sense of the system and the existence of speeds $\mathscr{U}$ such that $|\mathscr{U}| < 1$ since $c_\alpha f^{\alpha\beta} c_\beta > 0$ always. The condition $\det \bar{H} = 0$ cannot in general be solved algebraically for $\mathscr{U}^2$ because of the general anisotropy of the medium. In particular, as in general crystal structures the polarization of wave fronts is not automatically longitudinal or transverse. However, if we consider that we know *a priori* an amplitude direction, i.e. the wave-front polarization, then the corresponding speed follows from (3.25). For instance, for a *longitudinal* wave front LW such that $\bar{U}_\sigma = U_\parallel \lambda_\sigma$ and for an *a priori* transversely polarized wave front TW such that $\bar{U}_\sigma = U_\perp m_\sigma$ with $m_\sigma P^{\sigma\alpha} m_\alpha = 1$, $P^{\alpha\beta}\lambda_\alpha m_\beta = 0$, we immediately have the wave speeds from

LW: $$\mathscr{U}^2 = \mathscr{U}_\parallel^2 = \bar{Q}_\parallel(\lambda; S^+)/\rho^+ f_\parallel(\lambda, S^+), \tag{3.28}$$

and

TW: $$\mathscr{U}^2 = \mathscr{U}_\perp^2 = \bar{Q}_\perp(m, \lambda; S^+)/\rho^+ f_\perp(m; S^+), \tag{3.29}$$

respectively, with
$$\left.\begin{aligned} \bar{Q}_\parallel &= \lambda_\alpha \bar{Q}^{\alpha\beta}(\lambda; S^+)\,\lambda_\beta, \quad f_\parallel = \lambda_\alpha f^{\alpha\beta}\lambda_\beta, \\ \bar{Q}_\perp &= m_\alpha \bar{Q}^{\alpha\beta}(\lambda; S^+)\,m_\beta, \quad f_\perp = m_\alpha f^{\alpha\beta} m_\beta. \end{aligned}\right\} \tag{3.30}$$

4. The equation in perfectly conducting elastic bodies

The presence of a finite magnetic field in a perfectly conducting elastic body greatly complicates the wave propagation problem. A typical state or solution (3.1) now is symbolically replaced by

$$\hat{S} = S + \{\mathscr{H}\}, \tag{4.1}$$

where $\mathscr{H}$ satisfies Maxwell's equations. For a weak-discontinuity front W, $\nabla\mathscr{H}$ is discontinuous across W and, similarly to (3.4*a, b*) we set

$$[\![\nabla_\beta \mathscr{H}^\alpha]\!] = N_\beta \bar{H}^\alpha, \quad \bar{H}^\alpha = [\![\mathrm{D}_N \mathscr{H}^\alpha]\!] \tag{4.2}$$

at W. It proves useful to find the expression of $\bar{H}^\alpha$ in terms of $\bar{U}^\alpha$: applying the operator D_u to the second of (2.29) and taking the jump of the resulting equation across W, we obtain

$$\bar{H}^\alpha u_\alpha = -\mathscr{H}^\alpha \bar{U}_\alpha. \tag{4.3}$$

Taking now the jump of (2.26) we have

$$P^\alpha_{\cdot\lambda}\bar{H}^\lambda = N_0^{-1}[\mathscr{H}^\alpha(N_\gamma^\perp \bar{U}^\gamma) - (\mathscr{H}^\gamma N_\gamma^\perp)\,\bar{U}^\alpha]. \tag{4.4}$$

Then $$\begin{aligned} \bar{H}^\alpha &= -(\bar{H}^\gamma u_\gamma)\,u^\alpha + P^\alpha_{\cdot\gamma}\bar{H}^\lambda \\ &= (\mathscr{H}^\gamma \bar{U}_\gamma)\,u^\alpha + N_0^{-1}[\mathscr{H}^\alpha(N_\gamma^\perp \bar{U}^\gamma) - (\mathscr{H}^\gamma N_\gamma^\perp)\,\bar{U}^\alpha], \end{aligned} \tag{4.5}$$

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which is the desired result. Remark that in contrast with $\bar{U}^\alpha$ the four-vector $\bar{H}^\alpha$ is *not* purely spatial in general. However, on taking the jump of (2.25) we also have

$$N_\alpha \bar{H}^\alpha = N_0 u^\beta \bar{H}_\beta, \tag{4.6}$$

which shows that

$$\bar{H}^\alpha \lambda_\alpha = 0, \tag{4.7}$$

so that the infinitesimal discontinuity of the magnetic field is purely *transverse*, which, of course, is not surprising since it is an extreme case of electromagnetic waves.

The magnetoelastic wave problem now is reduced to accounting for the jump of (2.21), i.e.

$$\begin{aligned}[\![\tilde{\mathfrak{M}}^\alpha]\!] = \rho \tilde{f}^{\alpha}_{\cdot\beta}[\![\mathrm{D}_u u^\beta]\!] - P^{\alpha}_{\cdot\gamma} P^{\beta\lambda}[\![\nabla_\lambda t_\beta{}^\gamma]\!] \\ -\mu\{\mathscr{H}^\alpha[\![\nabla_\beta \mathscr{H}^\beta]\!] + \mathscr{H}^\beta P^{\alpha}_{\cdot\lambda}[\![\nabla_\beta \mathscr{H}^\lambda]\!] - P^{\alpha\lambda}\mathscr{H}_\gamma[\![\nabla_\lambda \mathscr{H}^\gamma]\!]\} = 0.\end{aligned} \tag{4.8}$$

That is, all other things being unchanged, we only have to replace $f^{\alpha\beta}$ by $\tilde{f}^{\alpha\beta}$ and to account for the last contribution in (4.8) on account of (4.5). A short calculation allows one to show that this contribution reads

$$-\mathfrak{M}^\alpha_{\mathrm{M}} = -[Q^{\alpha\sigma}_{\mathrm{M}}(\boldsymbol{\lambda};\, S^+)\,(1+N_0^2)/N_0 + N_0 \mu \mathscr{H}^\alpha \mathscr{H}^\sigma]\,\bar{U}_\sigma, \tag{4.9}$$

where the magnetic contribution to the total spatial Christoffel acoustic tensor has been defined by (cf. (3.20))

$$Q^{\alpha\sigma}_{\mathrm{M}}(\boldsymbol{\lambda};\, S^+) = C^{\beta\alpha\mu\sigma}_{\mathrm{M}}(S^+)\,\lambda_\beta \lambda_\mu = Q^{\sigma\alpha}_{\mathrm{M}}, \tag{4.10}$$

with

$$C^{\beta\alpha\mu\sigma}_{\mathrm{M}}(S^+) = \mu[\mathscr{H}^\mu \mathscr{H}^\beta P^{\alpha\sigma} - P^{\beta\alpha}(\mathscr{H}^\mu \mathscr{H}^\sigma - \mathscr{H}^2 P^{\mu\sigma}) - \mathscr{H}^\beta \mathscr{H}^\alpha P^{\mu\sigma}]\,(S^+). \tag{4.11}$$

The latter operator is *not* Hookean. However, the symmetry of $\boldsymbol{Q}_{\mathrm{M}}$ is obvious when the latter is written in intrinsic notation as

$$\boldsymbol{Q}_{\mathrm{M}} = \mu\{\mathscr{H}^2 \boldsymbol{\lambda}\otimes\boldsymbol{\lambda} + (\mathscr{H}\cdot\boldsymbol{\lambda})^2 \boldsymbol{P} - (\mathscr{H}\cdot\boldsymbol{\lambda})\,[(\boldsymbol{\lambda}\otimes\mathscr{H}) + (\boldsymbol{\lambda}\otimes\mathscr{H})^{\mathrm{T}}]\}. \tag{4.12}$$

With the new contribution (4.9) taken into account, the wave equation (3.25) is replaced by the following

$$\hat{H}^{\alpha\sigma}(\mathscr{U}^2, \boldsymbol{\lambda};\, S^+)\,\bar{U}_\sigma = 0, \tag{4.13}$$

where

$$\hat{H}^{\alpha\sigma} = \rho \hat{f}^{\alpha\sigma}(S^+)\,\mathscr{U}^2 - \hat{Q}^{\alpha\sigma}(\boldsymbol{\lambda};\, S^+) = \hat{H}^{\sigma\alpha} \tag{4.14}$$

and

$$\hat{Q}^{\alpha\sigma} = \bar{Q}^{\alpha\sigma} + Q^{\alpha\sigma}_{\mathrm{M}}, \quad \hat{f}^{\alpha\sigma} = \tilde{f}^{\alpha\sigma} - (\mu \mathscr{H}^\alpha \mathscr{H}^\sigma/\rho). \tag{4.15a, b}$$

The latter can also be written as

$$\hat{f}^{\alpha\sigma} = f^{\alpha\sigma} + f^{\alpha\sigma}_{\mathrm{M}}, \quad f^{\alpha\sigma}_{\mathrm{M}} \equiv (\mu/\rho)\,(\mathscr{H}^2 P^{\alpha\sigma} - \mathscr{H}^\alpha \mathscr{H}^\sigma). \tag{4.16}$$

The decompositions given in (4.15*a*) and (4.16) allow one to distinguish between the purely thermoelastic contributions and the magnetic ones. The *additive* character of these contributions is emphasized. By the same token (4.11) can be rewritten as

$$C^{\beta\alpha\mu\sigma}_{\mathrm{M}} = \rho f^{\mu\sigma}_{\mathrm{M}} P^{\beta\alpha} + \mu(\mathscr{H}^\beta \mathscr{H}^\mu P^{\alpha\sigma} - \mathscr{H}^\beta \mathscr{H}^\alpha P^{\mu\sigma}). \tag{4.17}$$

Again, a *strong ellipticity* condition of the type (cf. (3.27)

$$c_\alpha \hat{Q}^{\alpha\sigma} c_\sigma > 0, \tag{4.18}$$

will ensure the hyperbolicity in Hadamard's sense of the system and the existence of speeds $\mathscr{U}$ such that $|\mathscr{U}| < 1$ if $f^{\alpha\sigma}$ satisfies also an inequality of the type (4.18). Now we can prove the following *general* statement.

THEOREM. *If infinitesimal-discontinuity wave fronts can propagate in the purely thermoelastic case, then they can propagate in the magnetoelastic case also.*

Indeed, assume that in the thermoelastic case

$$c_\alpha \bar{Q}^{\alpha\sigma} c_\sigma > 0, \quad c_\alpha f^{\alpha\sigma} c_\sigma > 0. \tag{4.19}$$

It is immediately checked in intrinsic notation that

$$c_\alpha Q_{\mathrm{M}}^{\alpha\sigma} c_\sigma = \mu\{[\mathscr{H} - (\mathscr{H}\cdot\boldsymbol{\lambda})\,\mathbf{c}]^2 + 2(\mathscr{H}\cdot\boldsymbol{\lambda})\,(\mathscr{H}\cdot\mathbf{c})\,[1-\mathbf{c}\cdot\boldsymbol{\lambda}]\} > 0. \tag{4.20}$$

and

$$c_\alpha f_{\mathrm{M}}^{\alpha\sigma} c_\sigma = (\mu/\rho)\,[\mathscr{H}^2 - (\mathscr{H}\cdot\mathbf{c})^2] \geqslant 0, \tag{4.21}$$

so that the magnetic field can only reinforce the conditions (4.19). The proof of the theorem follows. In other words, as already proven in classical isotropic magnetoelasticity (Bazer & Ericson 1974) and for one-dimensional linear wave motion in relativistic isotropic magnetoelasticity (Maugin 1979*a*), the magnetic field *always* reinforces Hadamard's ellipticity condition (4.18).

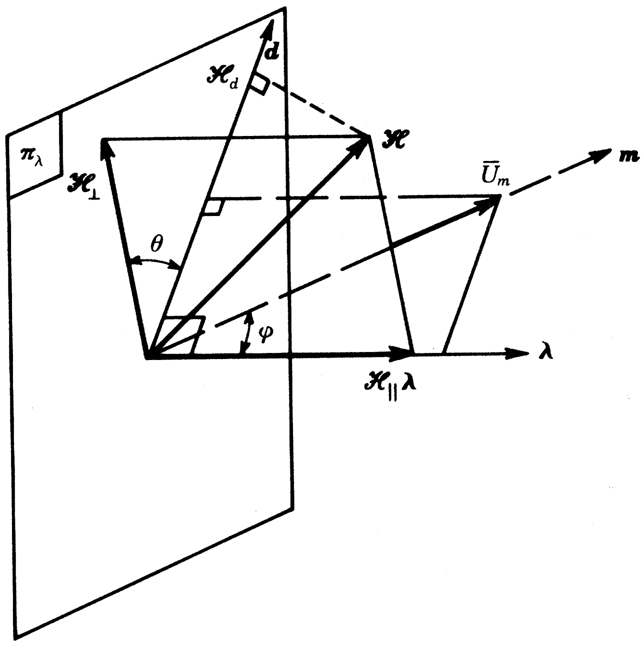

FIGURE 1.

We shall content ourselves with determining the propagation speed of *a priori* polarized wave fronts. We thus assume that we know an amplitude direction, for example

$$\bar{U}^\alpha = \bar{U}_{\mathrm{m}} m^\alpha, \quad m^\alpha u_\alpha = 0, \quad m_\alpha P^{\alpha\beta} m_\beta = 1, \quad \bar{U}_{\mathrm{m}} \neq 0. \tag{4.22}$$

In general $\bar{U}^\alpha$ is neither purely longitudinal nor purely transverse. Let φ be the angle made by he spatial unit four-vector $\boldsymbol{m}$ with $\boldsymbol{\lambda}$, let $\mathscr{H}_\parallel$ and $\mathscr{H}_\perp$ be the components of $\mathscr{H}(S^+)$ along $\boldsymbol{\lambda}$ and onto the two-plane π_λ orthogonal to $\boldsymbol{\lambda}$, and let θ be the angle made by $\mathscr{H}_\perp$ with the projection of $\boldsymbol{m}$ onto π_λ. The direction of latter projection is given by the unit spatial four-vector $\boldsymbol{d}$. We denote by index d a projection on this direction (cf. figure 1). We have thus

$$\bar{U}^\alpha = \bar{U}_{\mathrm{m}}(\cos\varphi\lambda^\alpha + \sin\varphi d^\alpha), \quad \lambda_\alpha P^{\alpha\beta} d_\beta = 0. \tag{4.23}$$

Let $\mathscr{U}_{\rm m}$ be the positive intrinsic speed of W corresponding to the amplitude $\bar{U}_{\rm m}$, if the latter is indeed an allowed solution of (4.13). On taking the inner product of (4.13) with $\bar{U}_\alpha$ and considering an amplitude as given by (4.23), we obtain

$$\hat{H}_\parallel \cos^2\varphi + \hat{H}_d \sin^2\varphi + 2\hat{H}_{(\parallel,d)} \sin\varphi\cos\varphi = 0 \tag{4.24}$$

with
$$\hat{H}_\parallel = \lambda_\alpha \hat{H}^{\alpha\sigma}\lambda_\sigma, \quad \hat{H}_d = d_\alpha \hat{H}^{\alpha\sigma} d_\sigma, \quad \hat{H}_{(\parallel,d)} = \lambda_\alpha \hat{H}^{\alpha\sigma} d_\sigma, \tag{4.25}$$

$$\hat{H}_\parallel = \rho \hat{f}_\parallel \mathscr{U}_{\rm m}^2 - \hat{Q}_\parallel = \rho\bar{f}_\parallel [1 + \mathscr{A}^2_{(\perp,\parallel)}]\, \mathscr{U}_{\rm m}^2 - [\bar{Q}_\parallel + (\boldsymbol{Q}_{\rm M})_\parallel], \tag{4.26}$$

$$\hat{H}_d = \rho \hat{f}_d \mathscr{U}_{\rm m}^2 - \hat{Q}_d = \rho\bar{f}_d [1 + \mathscr{A}^2_{(\parallel,d)} + \mathscr{A}^2_{(\perp,d)}(1-\cos^2\theta)]\, \mathscr{U}_{\rm m}^2 - [\bar{Q}_d + (\boldsymbol{Q}_{\rm M})_d], \tag{4.27}$$

$$\hat{H}_{(\parallel,d)} = \rho \hat{f}_{(\parallel,d)} \mathscr{U}_{\rm m}^2 - \hat{Q}_{(\parallel,d)}, \tag{4.28}$$

where
$$\left.\begin{aligned} \mathscr{A}^2_{(\perp,\parallel)} &= (\rho\bar{f}_\parallel)^{-1}\mu(\mathscr{H}^2 - \mathscr{H}_\parallel^2) = (\rho\bar{f}_\parallel)^{-1}\mu\mathscr{H}_\perp^2, \\ \rho\hat{f}_d &= \rho\bar{f}_d[1 + (\rho\bar{f}_d)^{-1}\mu(\mathscr{H}^2 - \mathscr{H}_d^2)]. \end{aligned}\right\} \tag{4.29}$$

But
$$\begin{aligned} \mu(\mathscr{H}^2 - \mathscr{H}_d^2) &= \mu(\mathscr{H}_\parallel^2 + \mathscr{H}_\perp^2 - \mathscr{H}_d^2) \\ &= \mu[\mathscr{H}_\parallel^2 + \mathscr{H}_\perp^2(1-\cos^2\theta)], \end{aligned} \tag{4.30}$$

so that on defining
$$\left.\begin{aligned} \mathscr{A}^2_{(\parallel,d)} &= (\rho\bar{f}_d)^{-1}\mu\mathscr{H}_\parallel^2, \\ \mathscr{A}^2_{(\perp,d)} &= (\rho\bar{f}_d)^{-1}\mu\mathscr{H}_\perp^2(1-\cos^2\theta), \end{aligned}\right\} \tag{4.31}$$

we readily obtain the second expression given in (4.27). From (4.21), (4.22) and the second of (4.23), we have

$$(\boldsymbol{Q}_{\rm M})_\parallel = \mu\mathscr{H}_\perp^2, \quad (\boldsymbol{Q}_{\rm M})_d = \mu\mathscr{H}_\parallel^2 \tag{4.32}$$

and
$$\hat{f}_{(\parallel,d)} = -\rho^{-1}[t_{(\parallel,d)} + \mu\mathscr{H}_\parallel\mathscr{H}_d], \quad \hat{Q}_{(\parallel,d)} = \bar{Q}_{(\parallel,d)} - \mu\mathscr{H}_\parallel\mathscr{H}_d. \tag{4.33}$$

On account of (4.21) through (4.33), (4.24) then yields the following solution for $\mathscr{U}_{\rm m}^2$:

$$\begin{aligned} \mathscr{U}_{\rm m}^2 = {} & \{(\bar{Q}_\parallel + \mu\mathscr{H}_\perp^2)\cos^2\varphi + (\bar{Q}_d + \mu\mathscr{H}_\parallel^2)\sin^2\varphi + 2\sin\varphi\cos\varphi[\bar{Q}_{(\parallel,d)} - \mu\mathscr{H}_\parallel\mathscr{H}_d]\} \\ & \div \rho\{\bar{f}_\parallel[1 + \mathscr{A}^2_{(\perp,\parallel)}]\cos^2\varphi + \bar{f}_d[1 + \mathscr{A}^2_{(\parallel,d)} + \mathscr{A}^2_{(\perp,d)}(1-\cos^2\theta)]\sin^2\varphi \\ & - 2\rho^{-1}\sin\varphi\cos\varphi[t_{(\parallel,d)} + \mu\mathscr{H}_\parallel\mathscr{H}_d]\}. \end{aligned} \tag{4.34}$$

By the same token, on account of (4.23) and (4.5), the associated infinitesimal discontinuity in the magnetic field, which we denote by $\bar{H}_{\rm m}^\alpha$, is given by ($\bar{U}_{\rm m} \neq 0$)

$$\bar{H}_{\rm m}^\alpha/\bar{U}_{\rm m} = (\mathscr{H}_\parallel\cos\varphi + \mathscr{H}_d\sin\varphi)\,u^\alpha + \mathscr{U}_{\rm m}^{-1}(\mathscr{H}_\perp^\alpha\cos\varphi - \mathscr{H}_\parallel d^\alpha\sin\varphi), \tag{4.35}$$

where $\mathscr{U}_{\rm m}$ is the positive value given by (4.34).

Although they clearly emphasize the purely thermoelastic contributions (indicated with an overbar) and the magnetic contributions (via $\mathscr{H}$ and the various non-dimensional numbers $\mathscr{A}_{(.,.)}$, which play the role of *Alfvén numbers* in the relativistic framework), the expressions (4.34) and (4.35) are far too general to be of any direct practical interest. To get some idea about the pertinence of these results consider two special cases of polarization:

(i) *Elastically longitudinal wave front* (the wave fronts are always transverse from the magnetic viewpoint). In this case $m^\alpha = \lambda^\alpha$, $\varphi = 0$, so that (4.34) and (4.35) reduce to

$$\mathscr{U}_\parallel^2 = (\bar{Q}_\parallel + \mu\mathscr{H}_\perp^2)/\rho\bar{f}_\parallel[1 + \mathscr{A}^2_{(\perp,\parallel)}] \tag{4.36}$$

and
$$\bar{H}^\alpha/\bar{U} = \mathscr{H}_\parallel u^\alpha + \mathscr{U}_\parallel^{-1}\mathscr{H}_\perp^\alpha. \tag{4.37}$$

For an *orthogonal setting* of the magnetic field $\mathscr{H}^\alpha(S^+)$, $\mathscr{H}^\alpha \equiv \mathscr{H}_\perp^\alpha$, (4.36) does not simplify further and (4.37) shows that $\bar{H}^\alpha$ is spatial and parallel to $\mathscr{H}^\alpha$ in the two-plane π_λ. For a *longitudinal* *setting* of the magnetic field, $\mathscr{H}_\perp \equiv \mathbf{0}$, $\mathscr{H}^\alpha = \mathscr{H}_\parallel \lambda^\alpha$, (4.36) reduces to the purely elastic value obtained in Maugin (1979*b*), but $\bar{H}^\alpha$ is purely time-like and independent of the properties of propagation (i.e. it does not depend on the wave speed).

(ii) *Elastically transverse wave fronts*. In this case $m^\alpha = d^\alpha$, $\varphi = \pi/2$, so that (4.34) and (4.35) reduce to

$$\mathscr{U}_\perp^2 = (\bar{Q}_d + \mu\mathscr{H}_\parallel^2)/\rho\bar{f}_d[1 + \mathscr{A}^2_{(\parallel, d)} + \mathscr{A}^2_{(\perp, d)}(1-\cos^2\theta)] \tag{4.38}$$

and

$$\bar{H}^\alpha/\bar{U} = \mathscr{H}_d u^\alpha - \mathscr{U}_\perp^{-1}\mathscr{H}_\parallel d^\alpha. \tag{4.39}$$

For a *purely orthogonal setting* of the magnetic field, (4.38) and (4.39) reduce to

$$\mathscr{U}_\perp^2 = \bar{Q}_d/\rho\bar{f}_d[1 + \mathscr{A}^2_{(\perp, d)}(1-\cos^2\theta)], \quad \bar{H}^\alpha/\bar{U} = \mathscr{H}_\perp u^\alpha \cos\theta. \tag{4.40 a, b}$$

Equation (4.40*a*) further reduces to the purely elastic value obtained in Maugin (1979*b*) whenever $\bar{U}^\alpha$ is parallel to $\mathscr{H}^\alpha$ in π_λ, while $\bar{H}^\alpha$ is always purely time-like and does not depend on the propagation speed, but it still depends on the angle made by the polarization vector and the direction of the initial magnetic field. In particular, $\bar{H}^\alpha$ vanishes whenever $\bar{U}^\alpha$ is orthogonal to the direction of the initial magnetic field. For a *purely longitudinal setting* of the initial magnetic field, (4.38) and (4.39) reduce to

$$\mathscr{U}_\perp^2 = (\bar{Q}_d + \mu\mathscr{H}_\parallel^2)/\rho\bar{f}_d[1 + \mathscr{A}^2_{(\parallel, d)}], \quad \bar{H}^\alpha/\bar{U} = -\mathscr{U}_\perp^{-1} d^\alpha \mathscr{H}_\parallel. \tag{4.41 a, b}$$

Hence $\bar{H}^\alpha$ is antiparallel with the elastic perturbation in this case. Note that the first of (4.41*a*) is a generalization of the concept of an *Alfvên wave front* considered in relativistic magnetohydrodynamics by Lichnerowicz (1971). Indeed, this Alfvén speed is given by (Lichnerowicz 1971, equation (12.6))

$$\mathscr{U}_A^2 = \mu\mathscr{H}_\parallel^2/\rho f(1 + \mathscr{A}_\parallel^2), \quad \mathscr{A}_\parallel^2 = \mu\mathscr{H}_\parallel^2/\rho, \tag{4.42}$$

where f is the scalar index of the *fluid* (for a fluid $\bar{Q}_d = 0$ and $\bar{f}_d$ reduces to $f = 1 + \epsilon + p/\rho$ where p is the pressure.

We finally note that in (4.36), (4.38) and (4.41*a*) the magnetic field provides an essentially *classical* contribution to the numerator which increases (stiffens) the value of the corresponding propagation speed as compared with that of the purely elastic case, whereas it contributes to the denominator, via Alfvén numbers, in *relativistic* corrections, and decreases the value of the corresponding propagation speed as compared with that of the purely elastic case. We shall later examine the extent to which the presence of a strong initial magnetic field does or does not favour the formation of *shock fronts* from weak magnetoelastic waves. To shed light on this difficult problem we first examine the purely elastic case.

5. Ray theory in the purely elastic case

To avoid increasing unnecessarily the already cumbersome nature of the subsequent computations we shall limit ourselves to the case of a propagation throughout a state S^+ that corresponds to *spatially homogeneously strained* and *stressed* bodies. That is

$$\left.\begin{aligned} (\nabla_\beta t^{\mu\delta})^+ = 0, \quad (\nabla_\beta X^K_\alpha)^+ = 0, \quad (\nabla_\beta \mathscr{C}^{KL})^+ = 0, \\ (\nabla_\beta \rho)^+ = 0, \quad (\nabla_\beta \eta)^+ = 0, \quad (\nabla_\beta \theta)^+ = 0. \end{aligned}\right\} \tag{5.1}$$

If we note that in general

$$[\![AB]\!] = [\![A]\!]\, B^+ + A^+[\![B]\!] - [\![A]\!]\,[\![B]\!] \tag{5.2}$$

if A and B suffer discontinuity jumps across W, then if $A = \nabla a \in S$ and $B = \nabla b \in S$, equations (5.1) imply that (5.2) reduces to

$$[\![(\nabla a)\,(\nabla b)]\!] = -[\![\nabla a]\!]\,[\![\nabla b]\!]. \tag{5.3}$$

This relation will be of constant use in the following development.

In contradistinction with the qualitative method used in Maugin (1977), we consider the following direct method to construct relatively easily the equation of rays (or bicharacteristics) associated with the system (2.13)–(2.16) of balance laws and the wave equation (3.25). We claim that the ray equation or, equivalently, the equation that governs the evolution of the amplitude $\bar{U}$ along the rays associated with (3.25) is none other than the equation.

$$\mathscr{R}^\alpha(\mathscr{U}^2, \bar{U};\, S^+, G_2^W) = [\![\mathscr{G}_\perp^\alpha]\!] = 0, \tag{5.4}$$

where G_2^W denotes the local geometry of W at the second order (i.e. it involves the curvature of W and tangential derivatives on W). Indeed, on account of (2.13) and (2.16), it is immediately found that

$$\begin{aligned}\mathscr{G}_\perp^\alpha &\equiv (\mathrm{D}_u \mathfrak{M}^\alpha)_\perp \\ &= \rho f^{\alpha\beta}\mathrm{D}_u^2 u_\beta - \rho f^{\alpha\beta}(\nabla_\gamma u^\gamma)\,\mathrm{D}_u u_\beta + P^{\alpha\beta}(t^{\mu\nu}\nabla_\mu u_\nu)\,\mathrm{D}_u u_\beta \\ &\quad - P^\alpha_{\cdot\beta}(\mathrm{D}_u t^{\gamma\beta})\,\mathrm{D}_u u_\gamma - (\nabla_\gamma u^\gamma)\, t^{\alpha\beta}\mathrm{D}_u u_\beta - P^\alpha_{\cdot\beta} D_u(P^\beta_{\cdot\gamma} P^\lambda_{\cdot\mu}\nabla_\lambda t^{\mu\gamma}),\end{aligned} \tag{5.5}$$

where we accounted for the fact that $(\mathrm{D}_u P^{\alpha\beta})_\perp \equiv 0$ and $\mathrm{D}_u^2 A = \mathrm{D}_u(\mathrm{D}_u A)$. Furthermore, noting that

$$u_\mu \nabla_\lambda t^{\beta\mu} = -t^{\beta\mu}\nabla_\lambda u_\mu, \quad u_\gamma u_\mu \nabla_\lambda t^{\gamma\mu} = 0, \quad u_\gamma u_\mu \mathrm{D}_u t^{\gamma\mu} = 0, \tag{5.6}$$

by virtue of the spatial nature of $\boldsymbol{t}$, we evaluate the last term in (5.5) as

$$\begin{aligned}P^\alpha_{\cdot\beta}\mathrm{D}_u(P^\beta_{\cdot\gamma} P^\lambda_{\cdot\mu}\nabla_\lambda t^{\mu\gamma}) &= P^\alpha_{\cdot\gamma} P^\lambda_{\cdot\mu}(\mathrm{D}_u \nabla_\lambda t^{\mu\gamma}) - t^{\mu\alpha}(\nabla_\lambda u_\mu)\,\mathrm{D}_u u^\lambda \\ &\quad + (\mathrm{D}_u t^{\alpha\beta})\,\mathrm{D}_u u_\beta - t^{\lambda\gamma}(\nabla_\lambda u_\gamma)\,\mathrm{D}_u u^\alpha.\end{aligned} \tag{5.7}$$

On substituting the result (5.7) into (5.5), noting that $\mathscr{R}^\alpha = [\![\mathscr{G}_\perp^\alpha]\!]$, using the property (5.3) and rearranging indices, we arrive at the equation

$$\begin{aligned}\mathscr{R}^\alpha &= \rho f^\alpha_{\cdot\beta}[\![\mathrm{D}_u^2 u^\beta]\!] + \{(\rho f^{\gamma\delta}P^{\alpha\beta} - t^{\gamma\delta}P^{\alpha\beta} - t^{\delta\alpha}P^{\gamma\beta})\,[\![\nabla_\gamma u_\delta]\!] \\ &\quad + 2[\![\mathrm{D}_u t^{\alpha\beta}]\!]_\perp\}\,[\![\mathrm{D}_u u_\beta]\!] - P^\alpha_{\cdot\gamma} P^\lambda_{\cdot\beta}[\![\mathrm{D}_u(\nabla_\lambda t^{\beta\gamma})]\!] = 0,\end{aligned} \tag{5.8}$$

where the quantities outside the open square brackets are to be evaluated at S^+.

The evaluation of the second term in (5.8) involves no difficulties. That of the first and third terms requires the evaluation of the value of jump discontinuities in second-order derivatives of the state S. For that purpose we shall rely heavily on the results of Maugin (1976). The following notation and definitions are needed. In terms of the Gaussian representation (3.3) of W we set

$$\left.\begin{aligned}\phi^\alpha_\Gamma &= \partial\phi^\alpha/\partial a^\Gamma, & \phi^\Gamma_\lambda &= \gamma^{\Gamma\Delta} g_{\lambda\mu}\phi^\mu_\Delta, \\ \gamma_{\Gamma\Delta} &= g_{\alpha\beta}\phi^\alpha_\Gamma \phi^\beta_\Delta, & b_{\Gamma\Delta} &= -\phi^\mu_\Gamma \phi^\gamma_\Delta \nabla_\gamma N_\mu,\end{aligned}\right\} \tag{5.9}$$

where $\boldsymbol{\gamma}$ is the local metric on W and $\boldsymbol{b}$ is its second fundamental form. We further set

$$\mathscr{U}^\alpha = [\![\mathrm{D}_N^2 u^\alpha]\!], \quad \mathscr{D} = \mathrm{D}_u + N_0 \mathrm{D}_N, \quad \mathrm{D}_N = N^\alpha \nabla_\alpha, \quad \mathrm{D}_N^2 A = \mathrm{D}_N(\mathrm{D}_N A), \tag{5.10}$$

where $\mathscr{D}$ is the invariant derivative following the motion of W along its normal in M. Then, in accordance with theorem 4.7 and corollary 4.12 in Maugin (1976), across W we have

$$[\![\mathrm{D}_u^2 u^\alpha]\!] = \mathscr{U}^\alpha N_0^2 - 2N_0 \mathscr{D}\bar{U}^\alpha - \bar{U}^\alpha \mathscr{D} N_0, \tag{5.11}$$

and

$$[\![\nabla_\lambda \nabla_\rho u^\beta]\!] = \mathscr{U}^\beta N_\lambda N_\rho + (N_\lambda \phi^\Gamma_\rho + N_\rho \phi^\Gamma_\lambda)\,\bar{U}^\beta_{.|\Gamma} - b_{\Gamma\Delta}\phi^\Gamma_\rho \phi^\Delta_\lambda \bar{U}^\beta, \tag{5.12}$$

where a vertical bar denotes covariant differentiation with respect to $\{a^\Gamma\}$. Clearly, in view of the last contribution in (5.8) we shall also need the jump of second-order derivatives of η, ρ and X^K_α. For the first of these, since across W

$$[\![\eta]\!] = 0, \quad [\![\nabla_\lambda \eta]\!] = 0, \quad [\![D_N \eta]\!] = 0, \tag{5.13}$$

and with $\mathscr{N} \equiv [\![D^2_N \eta]\!]$, an equation of the type of (5.12) for η reduces to

$$[\![\nabla_\gamma \nabla_\lambda \eta]\!] = \mathscr{N} N_\gamma N_\lambda. \tag{5.14}$$

But, on considering the equation $[\![D^2_u \eta]\!] = 0$ which follows from (2.14), we find that

$$[\![D^2_u \eta]\!] = -u^\lambda [\![\nabla_\lambda u^\gamma]\!] [\![\nabla_\gamma \eta]\!] + u^\gamma u^\lambda [\![\nabla_\lambda \nabla_\gamma \eta]\!] = 0 \tag{5.15}$$

on account of (5.3). Taking into account (5.13) and (5.14) we obtain that $\mathscr{N} = 0$ if and only if N_0 or $\mathscr{U}$ differs from zero. Hence $[\![\nabla_\gamma \nabla_\lambda \eta]\!] = 0$ in the following development. On taking the covariant derivative ∇_β of (2.13) we obtain

$$u^\lambda (\nabla_\beta \nabla_\lambda \rho) + (\nabla_\beta u^\lambda)(\nabla_\lambda \rho) + (\nabla_\beta \rho)(\nabla_\gamma u^\gamma) + \rho(\nabla_\beta \nabla_\gamma u^\gamma) = 0.$$

Taking the jump of this equation across W and taking into account (5.3) and (3.14) we have

$$u^\lambda [\![\nabla_\beta \nabla_\lambda \rho]\!] = 2\rho (N_\sigma \bar{U}^\sigma)^2 N_0^{-1} N_\beta - \rho [\![\nabla_\beta \nabla_\gamma u^\gamma]\!]. \tag{5.16}$$

Taking twice the covariant derivative of (2.29*a*) we obtain

$$(\nabla_\rho \nabla_\lambda u_\beta) X^{K\beta} + (\nabla_\lambda u_\beta)(\nabla_\rho X^{K\beta}) + (\nabla_\rho u^\beta)(\nabla_\lambda X^{K\beta}) + u_\beta (\nabla_\rho \nabla_\lambda X^{K\beta}) = 0.$$

Taking the jump of this equation across W and taking into account (5.3) and (2.13), we have

$$u^\beta [\![\nabla_\rho \nabla_\lambda X^K_\beta]\!] = 2 X^K_\sigma \bar{U}^\sigma (\bar{U}^\beta N_\beta) N_0^{-1} N_\lambda N_\rho - X^K_\beta [\![\nabla_\rho \nabla_\lambda u^\beta]\!]. \tag{5.17}$$

Since $\boldsymbol{g} \in C^3(\mathscr{B})$ we also note that

$$u^\beta [\![\nabla_\rho \nabla_\lambda X^K_\beta]\!] = u^\beta [\![\nabla_\rho \nabla_\beta X^K_\lambda]\!]. \tag{5.18}$$

It is fortunate that only the discontinuities (5.11), (5.12), (5.16) and (5.18), which are entirely expressible in terms of $\mathscr{U}^\alpha$, $\bar{U}^\alpha$ and $\bar{U}^\alpha_{\cdot|\Gamma}$, are needed to arrive at the form taken explicitly by (5.8). We shall give only a few intermediary steps in this lengthy, but not essentially difficult, calculation. In particular, we have

$$[\![D_u t^{\alpha\beta}]\!]_\perp = \bar{C}^{\beta\alpha\sigma\mu} N^\perp_\mu \bar{U}_\sigma, \tag{5.19}$$

so that on account of (3.14), (5.11) and (5.19) equation (5.8) first yields

$$\begin{aligned} &2\rho N_0 f^{\alpha\beta} \mathscr{D} \bar{U}_\beta - \rho N_0^2 f^{\alpha\beta} \mathscr{U}_\beta + \rho f^{\alpha\beta} \bar{U}_\beta \mathscr{D} N_0 \\ &\quad + N_0 \bar{U}_\epsilon \bar{U}_\sigma N^\perp_\beta (\rho f^{\beta\sigma} P^{\alpha\epsilon} - t^{\beta\sigma} P^{\alpha\epsilon} - t^{\alpha\sigma} P^{\epsilon\beta} + 2\bar{C}^{\epsilon\alpha\sigma\beta}) + \mathscr{A}^\alpha = 0, \end{aligned} \tag{5.20}$$

with
$$\mathscr{A}^\alpha = P^\alpha_{\cdot\gamma} P^\lambda_{\cdot\mu} [\![D_u \nabla_\lambda t^{\mu\gamma}]\!] = P^\alpha_{\cdot\gamma} P^\lambda_{\cdot\mu} u^\rho [\![\nabla_\rho \nabla_\lambda t^{\beta\gamma}]\!]. \tag{5.21}$$

On account of (5.13) and the fact that $[\![\nabla_\gamma \nabla_\lambda \eta]\!] = 0$, we find that

$$\mathscr{A}^\alpha = -N_0^{-1} \bar{U}_\sigma \bar{U}_\epsilon N^\perp_\beta N^\perp_\mu N^\perp_\theta \tilde{\mathscr{C}}^{\beta\alpha\sigma\mu\epsilon\theta} + \tilde{\mathscr{A}}^\alpha, \tag{5.22}$$

where we have set

$$\begin{aligned} \tilde{\mathscr{C}}^{\beta\alpha\sigma\mu\epsilon\theta}(S^+) = \mathscr{C}_{\mathrm{E}}^{\beta\alpha\sigma\mu\epsilon\theta}(S^+) &+ 2[(C_{\mathrm{E}}^{\beta\alpha\sigma\theta} P^{\epsilon\mu} + C_{\mathrm{E}}^{\sigma\alpha\epsilon\mu} P^{\beta\theta} + C_{\mathrm{E}}^{\beta\sigma\epsilon\mu} P^{\alpha\theta} + \tfrac{1}{2} C_{\mathrm{E}}^{\beta\alpha\sigma\epsilon} P^{\mu\beta}) \\ &- (t^{\sigma\alpha} P^{\epsilon\theta} P^{\beta\mu} + t^{\sigma\beta} P^{\alpha\theta} P^{\epsilon\mu} + t^{\sigma\epsilon} P^{\beta\theta} P^{\alpha\mu})]_{(S^+)}, \end{aligned} \tag{5.23}$$

$$\begin{aligned} \mathscr{C}_{\mathrm{E}}^{\beta\alpha\sigma\mu\epsilon\theta}(S^+) &= \mathscr{C}_{\mathrm{E}}^{(\beta\alpha)(\sigma\mu)(\epsilon\theta)} = \mathscr{C}_{\mathrm{E}}^{\mu\sigma\beta\alpha\epsilon\theta} = \mathscr{C}_{\mathrm{E}}^{\epsilon\theta\sigma\mu\beta\alpha} \\ &= 8\left(\rho \frac{\partial^3 \bar{\varepsilon}}{\partial \mathscr{C}^{KL} \partial \mathscr{C}^{MN} \partial \mathscr{C}^{PQ}} X^{K\beta} X^{L\alpha} X^{M\sigma} X^{N\mu} X^{P\epsilon} X^{Q\theta}\right)_{(S^+)} \end{aligned} \tag{5.24}$$

and
$$\mathscr{A}^\alpha \equiv P^\alpha_{.\gamma} P^\lambda_{.\beta}\{\rho^{-1}u^\mu[\![\nabla_\mu \nabla_\lambda \rho]\!]\, t^{\beta\gamma} - 4\rho(\partial\tilde{e}/\partial\mathscr{C}^{KL})\, X^{(L|\gamma|}u^\mu[\![\nabla_\mu \nabla_\lambda X^{K)\beta}]\!] - 4\rho(\partial^2\tilde{e}/\partial\mathscr{C}^{KL}\,\partial\mathscr{C}^{MN})\, X^{K\beta}X^{L\gamma}u^\mu[\![\nabla_\mu \nabla_\lambda X^{(M}_\nu]\!]\, X^{N)\nu}\}. \tag{5.25}$$

$\mathscr{C}_{\mathrm{E}}$ is the spatial tensor of third-order adiabatic elasticities. Note, by the way, that whereas the wave-propagation condition and a static perturbation problem, such as the one treated in Maugin (1978*g*), involve only second-order elasticities, the ray theory involves third-order elasticities, hence the convexity of the stress–strain relation. On account of (5.16)–(5.18) and (5.12), it is shown that (5.25) yields

$$\tilde{\mathscr{A}}^\alpha = -\bar{C}^{\beta\alpha\sigma\mu}[2N_0^{-1}N^\perp_\beta N^\perp_\mu N^{\perp\epsilon}\bar{U}_\sigma\bar{U}_\epsilon + b_{\Lambda\Gamma}\phi_\beta\phi^\Gamma_\mu\bar{U}_\sigma - (N_\beta\phi^\Gamma_\mu + N_\mu\phi^\Gamma_\beta)\,\bar{U}_{\sigma|\Gamma} - \mathscr{U}_\sigma N^\perp_\beta N^\perp_\mu], \tag{5.26}$$

where $\bar{C}$ is the tensor defined in components by equation (3.21).

In gathering the expressions (5.22) and (5.26) in (5.20) it is convenient to rearrange the term quadratic in $\bar{U}$ so as to account for the wave equation (3.25). To that purpose we notice that the resulting quadratic term can be written in the following form

$$\tilde{\mathscr{A}}^\alpha(\bar{U}^2) = N^\perp_\beta\bar{U}_\sigma\bar{U}_\epsilon[-(1+N_0^2)\,\bar{H}^{\beta\sigma}P^{\alpha\epsilon} + N_0^2\mathscr{R}^{\beta\sigma\alpha\epsilon} + N^\perp_\mu N^\perp_\theta\mathscr{C}^{\beta\alpha\sigma\mu\epsilon\theta}], \tag{5.27}$$

where we have set
$$\mathscr{R}^{\beta\sigma\alpha\epsilon} = t^{\beta\sigma}P^{\alpha\epsilon} + t^{\alpha\sigma}P^{\epsilon\beta} - 2\bar{C}^{\epsilon\alpha\sigma\beta}, \tag{5.28}$$

and
$$\bar{\mathscr{C}}^{\beta\alpha\sigma\mu\epsilon\theta} = \tilde{\mathscr{C}}^{\beta\alpha\sigma\mu\epsilon\theta} - \bar{C}^{\beta\mu\sigma\theta}P^{\alpha\epsilon}. \tag{5.29}$$

The last spatial tensorial quantity is the tensor of *apparent* (or effective) adiabatic elastic moduli of the third order. Equation (5.27) further simplifies on account of (3.25). Making use of the relations (3.9) and (3.16), which yield

$$N_0 = \mathscr{U}/(1-\mathscr{U}^2)^{\frac{1}{2}}, \quad (1+N_0^2)^{\frac{1}{2}} = (1-\mathscr{U}^2)^{-\frac{1}{2}}, \tag{5.30}$$

we finally transform (5.20) in the desired form:

$$2\mathscr{U}^2(\rho f^\alpha_{.\beta})_{(S^+)}\mathscr{D}\bar{U}^\beta + \mathscr{P}^{\alpha\sigma}_{(1)}(\mathscr{U}^2;\, S^+, G^W_2)\,\bar{U}_\sigma + \mathscr{P}^{\alpha\sigma\epsilon}_{(2)}(\mathscr{U}^2;\, S^+)\,\bar{U}_\sigma\bar{U}_\epsilon + \mathscr{P}^{\alpha\sigma\Gamma}_{(W)}(\mathscr{U}^2;\, S^+)\,\bar{U}_{\sigma|\Gamma} = \mathscr{U}(1-\mathscr{U}^2)^{-\frac{1}{2}}\bar{H}^{\alpha\sigma}\mathscr{U}_\sigma, \tag{5.31}$$

where
$$\mathscr{P}^{\alpha\sigma}_{(1)} = \mathscr{U}(1-\mathscr{U}^2)^{\frac{1}{2}}(\rho f^{\alpha\sigma}\mathscr{D}N_0 - \bar{C}^{\beta\alpha\mu\sigma}\phi^A_\beta\phi^\Gamma_\mu b_{\Lambda\Gamma}), \tag{5.32}$$

$$\mathscr{P}^{\alpha\sigma\epsilon}_{(2)} = -(1-\mathscr{U}^2)^{\frac{1}{2}}(\bar{\mathscr{C}}^{\beta\alpha\sigma\mu\epsilon\theta}\lambda_\beta\lambda_\mu\lambda_\theta + \mathscr{U}^2\mathscr{R}^{\beta\sigma\alpha\epsilon}\lambda_\beta), \tag{5.33}$$

$$\mathscr{P}^{\alpha\sigma\Gamma}_{(W)} = \mathscr{U}\bar{C}^{\beta\alpha\mu\sigma}(\lambda_\beta\phi^\Gamma_\mu + \lambda_\mu\phi^\Gamma_\beta), \tag{5.34}$$

and
$$\bar{\mathscr{C}}^{\beta\alpha\sigma\mu\epsilon\theta} = \mathscr{C}_{\mathrm{E}}^{\beta\alpha\sigma\mu\epsilon\theta} + 2(C_{\mathrm{E}}^{\beta\sigma\epsilon\mu}P^{\alpha\theta} + C_{\mathrm{E}}^{\alpha\sigma\epsilon\mu}P^{\beta\theta} + C_{\mathrm{E}}^{\beta\alpha\sigma\epsilon}P^{\mu\beta} + C_{\mathrm{E}}^{\beta\alpha\sigma\theta}P^{\epsilon\mu} + \bar{C}^{\beta\alpha\sigma\mu}P^{\epsilon\theta}) - [2(t^{\sigma\beta}P^{\alpha\theta}P^{\epsilon\mu} + t^{\sigma\epsilon}P^{\beta\theta}P^{\mu\alpha} + t^{\sigma\alpha}P^{\epsilon\theta}P^{\mu\beta} + \bar{C}^{\beta\mu\sigma\theta}P^{\alpha\epsilon} + C_{\mathrm{E}}^{\beta\alpha\sigma\epsilon}P^{\mu\beta}]. \tag{5.35}$$

Another way of writing the ray equation (5.31) is obtained by taking its inner product with $\bar{U}_\alpha$, from which on account of (3.25) follows the result

$$2\rho\mathscr{U}^2(\bar{U}_\alpha f^{\alpha\beta}\mathscr{D}\bar{U}_\beta) + \bar{U}_\alpha\,\mathscr{P}^{\alpha\sigma}_{(1)}\bar{U}_\sigma + \bar{U}_\alpha\,\mathscr{P}^{\alpha\sigma\epsilon}_{(2)}\bar{U}_\sigma\bar{U}_\epsilon + \bar{U}_\alpha\,\mathscr{P}^{\alpha\sigma\Gamma}_{(W)}\bar{U}_{\sigma|\Gamma} = 0, \tag{5.36}$$

where $\mathscr{U}^\alpha$ no longer intervenes.

Clearly, an equation such as (5.31) or (5.36) cannot be of great use in the general case. However, purely relativistic effects already appear in the formulation (5.31) in the form of factors involving $1-\mathscr{U}^2$, the presence of $f^{\alpha\beta}$ in place of $P^{\alpha\beta}$, and the relativistic nature of the term involving the tensor $\mathscr{R}$ in (5.33). If the introduction of this contribution is traced back in the analysis, it is immediately seen that the term arises from *inertial* effects contained in the tensorial index *f*. Equation (5.31) appears to be a relativistic generalization of a result due to Green (1967) in classical elasticity (see also the review by McCarthy (1975)). Great simplifications occur in (5.31) and (5.36) under the following conditions. Definitions of $\mathscr{D}$ that are equivalent to that given in (5.10) are (cf. Maugin 1976, equations (2.162) and (3.4))

$$\mathscr{D} = u_\alpha \mathrm{D}^\alpha_{\mathrm{T}} = \mathrm{d}_V, \tag{5.37}$$

where
$$\mathrm{D}^\alpha_{\mathrm{T}} = (g^{\alpha\beta} - N^\alpha N^\beta)\,\nabla_\beta, \quad \mathrm{d}_V f = f_{|\Gamma}\,\gamma^{\Gamma\Delta} u_\lambda \phi^\lambda_\Delta, \tag{5.38}$$

so that $\mathrm{D}^\alpha_{\mathrm{T}}$ is the tangential derivative on W. Then if the wave front is *flat* and propagates through a *homogeneously* strained and stressed state S^+, we have $\mathscr{D}N_0 = 0$, $b_{\Delta\Gamma} = 0$, and the second and fourth terms in both (5.31) and (5.36) vanish identically, so that we are left with

$$2\rho\mathscr{U}^2(\bar{U}_\alpha f^{\alpha\beta}\mathscr{D}\bar{U}_\beta) + \bar{U}_\alpha\, \mathscr{P}^{\alpha\sigma\epsilon}_{(2)}\bar{U}_\sigma \bar{U}_\epsilon = 0. \tag{5.39}$$

We shall exploit the ray equation only in these restricted conditions for which, symbolically,

$$\mathscr{D}\bar{U} = b\bar{U}^2. \tag{5.40}$$

The amplitude of the wave front will remain constant if and only if b vanishes identically.

6. The growth of *a priori* polarized flat elastic wave fronts

Let $\bar{U}_\alpha$ be in the spatial direction of the unit four-vector d_α, so that $\bar{U}_\alpha = U_d\, d_\alpha$, $d_\alpha u^\alpha = 0$, $P^{\alpha\beta} d_\alpha d_\beta = 1$. Furthermore, consider plane wave fronts and define a scalar parameter with the dimension of a length, σ (the distance along the spatial normal to W) such that $\sigma = \mathscr{U}_d \tau_{\mathrm{R}}$, where τ_{R} is the proper time in following W, i.e. $\mathscr{D} = \partial/\partial\tau_{\mathrm{R}}$. Then, with $U_d \neq 0$, (5.39) reduces to

$$\mathrm{d}U_d/\mathrm{d}\sigma - b_d\, U_d^2 = 0, \tag{6.1}$$

where we have set
$$b_d = -(d_\alpha\, \mathscr{P}^{\alpha\sigma\epsilon}_{(2)}\, d_\sigma d_\epsilon)/2\rho^+ f_d\, \mathscr{U}_d^3, \tag{6.2}$$

if $\mathscr{U}_d$ is the positive wave speed provided by (3.25), i.e. (cf. (3.28)–(3.29))

$$\mathscr{U}_d^2 = \frac{\bar{Q}_d(\boldsymbol{d}, \boldsymbol{\lambda};\, S^+)}{\rho^+ f_d(\boldsymbol{d};\, S^+)}, \quad \bar{Q}_d = d_\alpha \bar{Q}^{\alpha\beta}(\boldsymbol{\lambda};\, S^+)\, d_\beta, \quad f_d = d_\alpha f^{\alpha\beta}(S^+)\, d_\beta. \tag{6.3}$$

More precisely, on account of (5.33) and setting

$$b_d^* = \bar{\mathscr{C}}_d/2\rho^+\bar{c}_d^3, \quad \bar{c}_d^2 = \bar{Q}_d/\rho^+ \tag{6.4}$$

and
$$\bar{\mathscr{C}}_d = d_\alpha d_\sigma d_\epsilon \bar{\mathscr{C}}^{\beta\alpha\sigma\mu\epsilon\theta}\lambda_\beta\lambda_\mu\lambda_\theta, \quad \mathscr{R}_d = d_\sigma d_\alpha d_\epsilon \mathscr{R}^{\beta\alpha\sigma\epsilon}\lambda_\beta, \tag{6.5}$$

we can write
$$b_d = b_d^* \surd(f_d)\,(1-\mathscr{U}^2)^{\frac{1}{2}}[1+\mathscr{U}_d^2(\mathscr{R}_d/\bar{\mathscr{C}}_d)]. \tag{6.6}$$

Let U_d^0 be the initial value of the amplitude U_d. Then (6.1) integrates immediately to give the solution

$$U_d(\sigma) = U_d^0[1-(\sigma/\sigma_0)]^{-1}, \quad \sigma_0 = (U_d^0 b_d)^{-1}. \tag{6.7}$$

The wave front retains its strength (constant amplitude and same sign) if and only if $b_d = 0$, that is if $b_d^* = 0$ or $\bar{\mathscr{C}}_d = 0$. If $b_d^* \neq 0$, several cases must be considered according to whether the wave front is compressive or expansive and whether the effective third-order elasticity coefficient $\bar{\mathscr{C}}_d$ is positive or negative.

Equations (6.1) and (6.7) can be rewritten as

$$\mathrm{d}\,|U_d|/\mathrm{d}\sigma \mp b_d\,|U_d|^2 = 0, \quad |U_d(\sigma)| = U_d^0[1\mp(\sigma/\sigma_0)]^{-1}, \quad \sigma_0 = (|U_d^0|\, b_d)^{-1}, \tag{6.8}$$

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the minus sign corresponding to an expansive wave front ($U_d > 0$) and the plus sign to a compressive wave front ($U_d < 0$). For an *expansive* wave front $|U_d|$ becomes *unbounded* when $\sigma = \sigma_0$ with $\sigma_0 > 0$, hence if $b_d > 0$, or, essentially, $\overline{\mathscr{C}}_d > 0$; this means a *convex* apparent stress–strain curve at S^+. A strong discontinuity solution of the shock type ($[\![u^\alpha]\!] \neq 0$, $[\![\eta]\!] > 0$ across W) must be envisaged from then on. Otherwise $|U_d|$ decays as the wave front W traverses the material, and ultimately the infinitesimal discontinuity is smoothed out. For a *compressive* wave front ($U_d < 0$), $|U_d|$ becomes unbounded when $\sigma = -\sigma_0$ with $\sigma_0 < 0$, hence if $b_d < 0$ or, essentially, $\overline{\mathscr{C}}_d < 0$– this gives a *concave* apparent stress–strain curve at S^+. Otherwise $|U_d|$ decays to zero as the wave front travels throughout the material.

Let us now consider the influence of relativistic effects. These effects are clearly shown in (6.6). Since σ_0 behaves like b_d^{-1} according to the last of (6.8), we see that the factor $\sqrt{f_d}$, which represents a relativistic inertial effect, decreases the critical value of $|\sigma_0|$; the factor $(1-\mathscr{U}_d^2)^{\frac{1}{2}}$, a relativistic effect that involves only the behaviour of the elastic body at the second order, slightly increases this value; whereas the last factor within brackets in (6.6) will decrease this value if $\mathscr{R}_d$ and $\overline{\mathscr{C}}_d$ have the same sign, and will increase it if $\mathscr{R}_d$ and $\overline{\mathscr{C}}_d$ have opposite signs. In conclusion we see that the last effect, a combined relativity–material contribution, will favour the *strengthening* of a compressive wave front if $\mathscr{R}_d < 0$ and that of an expansive wave front if $\mathscr{R}_d > 0$. We shall illustrate in the remainder of this section the role played by both the initial state S^+ and the nonlinearity of the material, by considering a plausible simple elastic model.

Propagation in a quasi-Hookean body

If the elastic body whose general constitutive equations are given by (2.10) is isotropic, then ϵ depends on $\mathscr{C}$ only through the three elementary invariants $I_k = \operatorname{tr}\mathscr{C}^k$, $k = 1, 2, 3$, of the material tensor $\mathscr{C}$. On setting

$$\tilde{\epsilon}_k = \partial\tilde{\epsilon}/\partial I_k, \quad \tilde{\epsilon}_{ij} = \partial^2\tilde{\epsilon}/\partial I_i\,\partial I_j = \tilde{\epsilon}_{ji},$$

after a short computation we have

$$\begin{aligned}\partial^2\tilde{\epsilon}/\partial\mathscr{C}^{KL}\,\partial\mathscr{C}^{MN} = {} & \tilde{\epsilon}_{11}\delta_{KL}\delta_{MN} + 2\tilde{\epsilon}_2(\delta_{KM}\delta_{LN} + \delta_{ML}\delta_{KN}) + 2\tilde{\epsilon}_{12}\mathscr{C}_{MN}\delta_{KL}\\ & + 3\tilde{\epsilon}_3(\delta_{KM}\mathscr{C}_{LN} + \mathscr{C}_{KM}\delta_{LN}) + 2\tilde{\epsilon}_{12}\delta_{MN}\mathscr{C}_{KL}\\ & + 4\tilde{\epsilon}_{22}\mathscr{C}_{KL}\mathscr{C}_{MN} + 3\tilde{\epsilon}_{13}(\mathscr{C}_{MQ}\mathscr{C}^{Q}_{\cdot N}\delta_{KL} + \mathscr{C}_{KP}\mathscr{C}^{P}_{\cdot L}\delta_{MN})\\ & + 6\tilde{\epsilon}_{23}(\mathscr{C}_{\dot{M}}{}^{Q}\mathscr{C}_{QN}\mathscr{C}_{KL} + \mathscr{C}_{MN}\mathscr{C}_{KP}\mathscr{C}^{P}_{\cdot L})\\ & + 9\tilde{\epsilon}_{33}\mathscr{C}_{KP}\mathscr{C}^{P}_{\cdot L}\mathscr{C}_{MQ}\mathscr{C}_{\dot{N}}{}^{Q}.\end{aligned} \tag{6.9}$$

Let us assume for example that $|\mathscr{C}| = (\operatorname{tr}\mathscr{C}^2)^{\frac{1}{2}} \ll 1$. Then with (3.22) we can set

$$\begin{aligned}C_{\mathrm{E}}^{\beta\alpha\sigma\mu}(S^+) = {} & \hat{\lambda}(S^+)\,P^{\beta\alpha}(S^+)\,P^{\sigma\mu}(S^+)\\ & + \hat{\mu}(S^+)\,[P^{\beta\sigma}(S^+)\,P^{\alpha\mu}(S^+) + P^{\beta\mu}(S^+)\,P^{\alpha\sigma}(S^+)] + O(|\mathscr{C}(S^+)|),\end{aligned} \tag{6.10}$$

with
$$\hat{\lambda}(S^+) = 4\rho^+\tilde{\epsilon}_{11}, \quad \hat{\mu}(S^+) = 8\rho^+\tilde{\epsilon}_2, \tag{6.11}$$

where $\tilde{\epsilon}_{11}$ and $\tilde{\epsilon}_2$ are evaluated at $\mathscr{C} = \mathbf{0}$. Equation (6.10) represents the closest approximation to the Hookean elasticity tensor of an *isotropic* linear elastic body. The parameters $\hat{\lambda}$ and $\hat{\mu}$ are analogous to Lamé's coefficients. Consistently with (6.10), we shall take

$$\mathscr{C}_{\mathrm{E}}^{\beta\alpha\sigma\mu\epsilon\theta}(\mathrm{S}^+) = 0. \tag{6.12}$$

Furthermore, we shall assume that S^+ corresponds to a homogeneous state of high hydrostatic pressure p_0 such that

$$t^{\alpha\beta}(S^+) = -p_0 P^{\alpha\beta}(S^+). \tag{6.13}$$

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The model provided by (6.10), (6.12) and (6.13) is the simplest one that allows us to illustrate the effects of a simple, but physically interesting, initial state. In particular, (6.12) means that there are no adiabatic elasticities of the third order, and for one-dimensional states, the stress–strain relationship is linear. The strengthening of the wave front will therefore involve only the scalars $\hat{\lambda}$ and $\hat{\mu}$ and the pressure p_0. For further use we shall also introduce the following parameters:

$$\left.\begin{aligned} c_{\mathrm{L}}^2 &= (\hat{\lambda}+2\hat{\mu})/\rho^+, \quad c_{\mathrm{T}}^2 = \hat{\mu}/\rho^+, \quad \zeta_{\mathrm{LT}} = c_{\mathrm{L}}^2/c_{\mathrm{T}}^2, \\ \epsilon_p &= p_0/\rho^+ c_{\mathrm{L}}^2, \quad \epsilon_{\mathrm{T}} = p_0/\rho^+ c_{\mathrm{T}}^2 = 3\epsilon_p\,\zeta_{\mathrm{LT}}, \end{aligned}\right\} \tag{6.14}$$

of which the first two are the classical longitudinal and transverse elastic-disturbance speeds, and ϵ_p or ϵ_{T} is a non-dimensional parameter that accounts for the initial state p_0.

Before proceeding to the application of the scheme (6.10)–(6.13) to the growth of plane waves, we must notice that the *isotropic* hypothesis contained in (6.10) is somewhat in contradiction with introductory remarks. The results that follow will apply practically only if the elastic body of interest is *weakly elastically anisotropic*. By this we mean the following. If, for instance, the external crust of dense stars has a *cubic* structure then, theoretically, instead of (6.10), $\boldsymbol{C}_{\mathrm{E}}$ will have a representation of the form

$$C_{\mathrm{E}}^{\beta\alpha\sigma\mu}(S^+) = \xi(c_{11}-c_{12})\,P^{\beta\alpha\sigma\mu} + c_{12}P^{\beta\alpha}P^{\sigma\mu} + c_{44}(P^{\beta\sigma}P^{\alpha\mu} + P^{\beta\mu}P^{\alpha\sigma}), \tag{6.15}$$

where $P^{\beta\alpha\sigma\mu}$ is a spatial completely symmetric tensor that equals one if all indices are equal and zero otherwise, c_{11}, c_{12} and c_{44} are three elastic moduli and ξ is the non-dimensional parameter defined by

$$\xi = 1 - 2c_{44}/(c_{11}-c_{12}) \tag{6.16}$$

in crystallography (see, for example, Strauss 1968). The representation (6.15) can reasonably be replaced by the principal part of the expression (6.10) for practical purposes if and only if ξ is very small, in which case the cubic structure is said to be weakly elastically anisotropic. We shall assume that this is the case of the relativistic dense matter of interest.

With (6.10) accepted as such the possible wave modes that are solutions of (3.25) are necessarily either *longitudinal* or *transverse*, and the results (3.28), (3.29) immediately apply with

$$\mathscr{U}_{\parallel}^2 = c_{\mathrm{L}}^2(1+3\epsilon_p)f_0^{-1}, \quad \mathscr{U}_{\perp}^2 = c_{\mathrm{T}}^2(1+\epsilon_p\,\zeta_{\mathrm{LT}})f_0^{-1}, \tag{6.17}$$

where
$$f_0 = f_{\parallel} = f_{\perp} = 1+\epsilon(S^+) = p_0/\rho^+ = 1+\epsilon(S^+)/c^2+\epsilon_p\beta_{\mathrm{L}}^2, \tag{6.18}$$

in which we have set $\beta_{\mathrm{L}} = c_{\mathrm{L}}/c$, used the definitions (6.14), and written the last of (6.18) in dimensional units. Since the body practically behaves isotropically, the universal relation (5.26) established in Maugin (1978*d*) holds good *a priori*, so that the speed of sound in the state S^+, defined in agreement with Lichnerowicz, is given by

$$a^2(S^+) = \mathscr{U}_{\parallel}^2 - \tfrac{4}{3}\mathscr{U}_{\perp}^2 = (a_0^2+a_{\mathrm{f}}^2)f_0^{-1} \tag{6.19}$$

with
$$a_0^2 = B_0/\rho^+, \quad a_{\mathrm{f}}^2 = 5p_0/3\rho^+ = \tfrac{5}{3}\epsilon_p c_{\mathrm{L}}^2, \quad B_0 = \tfrac{1}{3}(3\hat{\lambda}+2\hat{\mu}), \tag{6.20}$$

where B_0 is the so-called bulk modulus of the material, a_0 is the speed of sound (without relativistic corrections), in the absence of initial pressure, and a_{f} happens to have the same definition as that of the *frozen sound speed* for diatomic molecules (as it occurs in the study of the influence of relaxation effects on shocks in gases of diatomic molecules (cf. Whitham 1974, p. 359).

The remaining parameters needed for the study of the growth of waves are obtained by replacing the index d either by $\parallel$ or by $\perp$ in the previously established equations, hence by replacing d_α successively by λ_α and m_α (with $m_\alpha P^{\alpha\beta}\lambda_\beta = 0$, $m_\alpha P^{\alpha\beta} m_\beta = 1$) in the defining equations. It is thus found that

$$\left.\begin{aligned} \overline{\mathscr{C}}_\parallel &= 8\rho^+ c_L^2(1+\tfrac{1}{8}9\epsilon_p) > 0, \quad \overline{\mathscr{C}}_\perp = 0,\\ \mathscr{R}_\parallel &= -2\rho^+ c_L^2(1+4\epsilon_p) < 0, \quad \mathscr{R}_\perp = 0. \end{aligned}\right\} \tag{6.21}$$

It immediately follows that $b_\perp = 0$ and *transverse infinitesimal discontinuities travel with constant amplitude throughout* S^+, whereas for longitudinal infinitesimal discontinuities of amplitude U (6.1) holds good with

$$b_\parallel = b^* f_0^{\frac{1}{2}}(1-\mathscr{U}_\parallel^2)^{\frac{1}{2}}\{1-\tfrac{1}{4}\mathscr{U}_\parallel^2[(1+4\epsilon_p)/(1+\tfrac{9}{8}\epsilon_p)]\} \tag{6.22}$$

and

$$b^* = \frac{\overline{\mathscr{C}}_\parallel}{2\rho^+ c_L^3(1+3\epsilon_p)^{\frac{3}{2}}} = \frac{4(1+\frac{9}{8}\epsilon_p)}{c_L(1+3\epsilon_p)^{\frac{3}{2}}}. \tag{6.23}$$

Since $\overline{\mathscr{C}}_\parallel$ and $\mathscr{R}_\parallel$ have opposite signs, the last factor in (6.22) will *not* favour the steepening of compressive wave fronts. This allows us to make a brief digression on *nonlinear elastic* bodies. Equation (6.12) essentially means that the body behaves linearly. However, if we retain the simple representation (6.10) for second-order elasticities, and (6.13) for the initial state, but we now consider $\mathscr{C}_E \neq \mathbf{0}$, then in the same circumstances $\mathscr{R}_\parallel$ retains its negative value, and the last contribution in the general equation (6.6) will indeed favour the steepening or strengthening of the wave front if and only if $(\mathscr{C}_E)_\parallel$ is 'sufficiently negative' (hence a stress–strain curve for one-dimensional motions in the λ-direction that is *markedly concave*) so that $\overline{\mathscr{C}}_\parallel$ and $\mathscr{R}_\parallel$ acquire the same, negative, sign. This occurs if and only if

$$(\mathscr{C}_E)_\parallel < 0, \quad |(\mathscr{C}_E)_\parallel| > 8\rho^+ c_L^2(1+\tfrac{9}{8}\epsilon_p). \tag{6.24 a, b}$$

Then in this *nonlinear* medium subjected to an initial state of homogeneous high hydrostatic pressure, compressive waves will ultimately form shocks via the phenomenon of formation of caustics due to focusing. Returning to the case represented by (6.22) we find that the exact critical value of σ is given by

$$\begin{aligned} \sigma_0 &= \tilde{\sigma}_0(c_L, |U_\parallel^0|, p_0)\\ &= \frac{c_L}{4|U_\parallel^0|}\frac{(1+3\epsilon_p)^{\frac{3}{2}} f_0^2}{(1+\frac{9}{8}\epsilon_p)[f_0-\beta_L^2(1+3\epsilon_p)]^{\frac{1}{2}}}\left\{f_0 - \frac{\beta_L^2(1+3\epsilon_p)(1+4\epsilon_p)}{(1+\frac{9}{8}\epsilon_p)}\right\}^{-1}, \end{aligned} \tag{6.25}$$

with

$$f_0 = 1+\epsilon/c^2+\epsilon_p\beta_L^2, \quad \beta_L = c_L/c, \quad \epsilon_p = p_0/\rho^+ c_L^2. \tag{6.26}$$

Both relativistic effects (via ϵ/c^2 and β_L) and the effects due to p_0 (via ϵ_p) are illustrated. The obtaining of the quantitative result (6.25) concludes the present section. For a weakly relativistic motion and if the initial state of hydrostatic pressure is sufficiently low, both parameters β_L and ϵ_p can be treated as infinitesimally small quantities, and the asymptotic behaviour of σ_0 follows immediately from (6.25).

7. Ray theory in the magnetoelastic case

In looking for the equation that governs the variations in the amplitude of the wave solution of (4.13), we follow the steps in the derivation given in §5, so we show here only how to obtain additional contributions. The general *ray equation* will be given by

$$\tilde{\mathscr{R}}^\alpha(\mathscr{U}^2, \overline{U}, S^+, G_2^W) = [\![\mathscr{G}^\alpha]\!]_\perp = [\![\mathscr{G}_\perp^\alpha]\!] = 0 \tag{7.1}$$

with

$$\tilde{\mathscr{G}}_\perp^\alpha = (\mathrm{D}_u \widetilde{\mathfrak{M}}^\alpha)_\perp, \tag{7.2}$$

where $\tilde{\mathfrak{M}}^{\alpha} = 0$ is the equation (2.21). In the calculation of (7.2) we particularly need the expression $\rho(\mathrm{D}_u \tilde{f}^{\alpha\beta})_{\perp}$. With $(\mathrm{D}_u P^{\alpha\beta})_{\perp} \equiv 0$ and (2.13) we first have

$$\rho(\mathrm{D}_u \tilde{f}^{\alpha\beta})_{\perp} = \rho P^{\alpha\beta}\mathrm{D}_u[1+\epsilon+(\mu\mathscr{H}^2/\rho)] + \rho^{-1}(\mathrm{D}_u\rho)\,t^{\alpha\beta} - (\mathrm{D}_u t^{\alpha\beta})_{\perp}$$
$$= [\rho\mathrm{D}_u(\epsilon+\tfrac{1}{2}\mu\mathscr{H}^2/\rho) + \tfrac{1}{2}\mathrm{D}_u\mu\mathscr{H}^2 + \tfrac{1}{2}\mu\mathscr{H}^2(\nabla_\gamma u^\gamma)]\,P^{\alpha\beta} - (\nabla_\gamma u^\gamma)\,t^{\alpha\beta} - (D_u t^{\alpha\beta})_{\perp}. \quad (7.3)$$

Taking account now of (2.22) and (2.27), we can evaluate the first two contributions in the first term in (7.3) to arrive at

$$\rho(\mathrm{D}_u \tilde{f}^{\alpha\beta})_{\perp} = (t^{\mu\nu}+2t^{\mu\nu}_{\mathrm{M}})\,(\nabla^{\perp}_{(\mu}u_{\nu)}) - (\nabla_\gamma u^\gamma)\,t^{\alpha\beta} - (\mathrm{D}_u t^{\alpha\beta})_{\perp}. \quad (7.4)$$

On collecting the partial results already obtained, in lieu of (5.5) we have

$$\tilde{\mathscr{G}}^{\alpha}_{\perp} = \rho\tilde{f}^{\alpha}_{\cdot\beta}\,\mathrm{D}^2_u u^\beta - \{[\rho\tilde{f}^{\alpha\beta}P^{\gamma\delta} - 2P^{\alpha\beta}(t^{\gamma\delta}+t^{\gamma\delta}_{\mathrm{M}}) + t^{\alpha\beta}P^{\gamma\delta} - t^{\alpha\delta}P^{\beta\gamma}]\,(\nabla_\gamma u_\delta) + 2(\mathrm{D}_u t^{\alpha\beta})_{\perp}\}\,(\mathrm{D}_u u_\beta)$$
$$- P^{\alpha}_{\cdot\gamma}P^{\lambda}_{\cdot\mu}\mathrm{D}_u(\nabla_\lambda t^{\mu\gamma}) + \mathscr{G}^{\alpha}_{\mathrm{M}} = 0, \quad (7.5)$$

where
$$\mathscr{G}^{\alpha}_{\mathrm{M}} = -\mu(\mathrm{D}_u[\mathscr{H}^\alpha\nabla_\beta\mathscr{H}^\beta + \mathscr{H}^\beta(\nabla_\beta\mathscr{H}^\alpha)_{\perp} - \mathscr{H}_\gamma\nabla^{\perp\alpha}\mathscr{H}^\gamma])_{\perp}. \quad (7.6)$$

Using the fact that

$$P^{\alpha}_{\cdot\beta}(\mathrm{D}_u P^{\beta\lambda}) = u^\lambda(\mathrm{D}_u u^\alpha), \quad (\nabla_\beta\mathscr{H}_\lambda)\,u^\lambda = -\mathscr{H}^\lambda\nabla_\beta u_\lambda, \quad (7.7)$$

we find that

$$\mathscr{G}^{\alpha}_{\mathrm{M}} = -\mu[(\mathrm{D}_u\mathscr{H}^\alpha)_{\perp}(\nabla_\beta\mathscr{H}^\beta) + \mathscr{H}^\alpha\mathrm{D}_u(\nabla_\beta\mathscr{H}^\beta) + (\mathrm{D}_u\mathscr{H}^\beta)\,(\nabla_\beta\mathscr{H}^\lambda)\,P^{\alpha}_{\cdot\lambda}$$
$$+\mathscr{H}^\beta\mathrm{D}_u(\nabla_\beta\mathscr{H}^\lambda)\,P^{\alpha}_{\cdot\lambda} - (\mathrm{D}_u\mathscr{H}_\gamma)\,(\nabla^{\perp\alpha}\mathscr{H}^\gamma) - P^{\alpha\lambda}\mathscr{H}_\gamma\mathrm{D}_u(\nabla_\lambda\mathscr{H}^\gamma)$$
$$-\mathscr{H}^\beta\mathscr{H}^\lambda(\nabla^{\perp}_\beta u_\lambda)\,(\mathrm{D}_u u^\alpha) - \mathscr{H}_\gamma(\mathrm{D}_u\mathscr{H}^\gamma)\,(\mathrm{D}_u u^\alpha)]. \quad (7.8)$$

The contributions in $\mathrm{D}_u(\nabla_\lambda t^{\mu\gamma})$ and $\mathrm{D}_u(\nabla_\beta\mathscr{H}^\lambda)$ which appear in (7.5) on account of (7.8) show that we shall have to deal with second-order derivatives of the fields and the associated jumps across W. With Reference to § 5 for the terms that involve $t^{\mu\gamma}$, it remains to evaluate the fields $\mathrm{D}_u(\nabla_\beta\mathscr{H}^\beta)$ and $[\mathrm{D}_u(\nabla_\lambda\mathscr{H}^\gamma)]_{\perp}$. This is achieved as follows. From (2.25) and (2.29*b*), we have

$$\mathrm{D}_u(\nabla_\alpha\mathscr{H}^\alpha) = -\mathrm{D}_u[u^\alpha(u^\beta\nabla_\alpha\mathscr{H}_\beta)] = \mathrm{D}_u(u^\alpha\mathscr{H}^\beta\nabla_\alpha u_\beta)$$
$$= (\mathrm{D}_u u_\beta)\,(\mathrm{D}_u\mathscr{H}^\beta) + \mathscr{H}_\beta\mathrm{D}^2_u u^\beta. \quad (7.9)$$

On taking the covariant derivative ∇_λ of (2.26) and accounting for the fact that

$$\nabla_\lambda(\mathrm{D}_u\mathscr{H}^\gamma) = u^\mu\nabla_\lambda\nabla_\alpha\mathscr{H}^\gamma + (\nabla_\lambda u^\mu)\,(\nabla_\alpha\mathscr{H}^\gamma) \quad (7.10)$$

and
$$\nabla_\lambda\nabla_\mu\mathscr{H}^\gamma = \nabla_\mu\nabla_\lambda\mathscr{H}^\gamma + R^{\gamma}_{\cdot\mu\lambda\rho}\mathscr{H}^\rho, \quad (7.11)$$

where $R^{\gamma}_{\cdot\mu\lambda\rho}$ is the Riemann–Christoffel curvature tensor based on $\boldsymbol{g}$, we obtain

$$P^{\beta}_{\cdot\gamma}\mathrm{D}_u(\nabla_\lambda\mathscr{H}^\gamma) = P^{\beta}_{\cdot\gamma}u^\mu R^{\gamma}_{\cdot\mu\lambda\rho}\mathscr{H}^\rho - (\nabla_\lambda u^\mu)\,(\nabla_\mu\mathscr{H}^\gamma)\,P^{\beta}_{\cdot\gamma} - (\nabla_\lambda\mathscr{H}^\beta)\,(\nabla_\alpha u^\alpha)$$
$$+(\nabla_\lambda\mathscr{H}^\alpha)\,(\nabla_\alpha u_\beta) - (\mathrm{D}_u\mathscr{H}^\gamma)\,u_\gamma(\nabla_\lambda u^\beta) - \mathscr{H}^\beta(\nabla_\lambda\nabla_\alpha u^\alpha)$$
$$+\mathscr{H}^\alpha(\nabla_\lambda\nabla_\alpha u_\gamma)\,P^{\beta}_{\cdot\gamma}. \quad (7.12)$$

By the same token we have the fully spatial result

$$[\mathrm{D}_u(\nabla_\lambda\mathscr{H}^\gamma)]_{\perp} = P^{\gamma}_{\cdot\xi}P^{\xi}_{\cdot\lambda}u^\mu R^{\xi}_{\cdot\mu\xi\rho}\mathscr{H}^\rho - (\nabla^{\perp}_\lambda u^\mu)\,(\nabla_\mu\mathscr{H}^\beta)\,P^{\gamma}_{\cdot\beta} - (\nabla_\lambda\mathscr{H}^\gamma)_{\perp}(\nabla_\alpha u^\alpha)$$
$$+(\nabla^{\perp}_\lambda\mathscr{H}^\alpha)\,(\nabla_\alpha u^\gamma) - (\mathrm{D}_u\mathscr{H}^\mu)\,u_\mu(\nabla_\lambda u^\gamma)_{\perp} - \mathscr{H}^\gamma(\nabla_\lambda\nabla_\alpha u^\alpha)$$
$$+\mathscr{H}^\alpha(\nabla^{\perp}_\lambda\nabla_\alpha u^\beta)\,P^{\gamma}_{\cdot\beta}. \quad (7.13)$$

On account of the results (7.9)–(7.13) all second-order derivatives of $\mathscr{H}$ can be eliminated from (7.8) by terms of second-order derivatives of $\boldsymbol{u}$. Generalizing the assumption made in § 5, we consider that the state S^+ ahead of W is now that of a spatially homogeneously strained, stressed and *magnetized* body. In addition to (5.1) we therefore have

$$(\nabla_\beta \mathscr{H}^\alpha)^+ = 0, \tag{7.14}$$

and use can be made of the simplified jump relation (5.3). We can deduce from (4.5) the following useful results

$$N_\beta \bar{H}^\beta = N_0 u^\beta \bar{H}_\beta = -N_0 (\mathscr{H}^\beta \bar{U}_\beta), \tag{7.15}$$

$$\bar{H}^\beta \bar{H}_\beta = -(\mathscr{H}^\gamma \bar{U}_\gamma)^2 + N_0^{-2}[\mathscr{H}^2 (N_\gamma^\perp \bar{U}^\gamma)^2 + (\mathscr{H}^\gamma N_\gamma^\perp)^2 \bar{U}^\alpha \bar{U}_\alpha - 2(N_\gamma^\perp \bar{U}^\gamma)(\mathscr{H}^\mu N_\mu^\perp)(\mathscr{H}^\zeta \bar{U}_\zeta)] \tag{7.16}$$

and
$$\bar{H}^\gamma \mathscr{H}_\gamma = N_0^{-1}[\mathscr{H}^2 (N_\mu^\perp \bar{U}^\mu) - (\mathscr{H}^\mu N_\mu^\perp)(\mathscr{H}_\gamma \bar{U}^\gamma)]. \tag{7.17}$$

On taking the jump of (7.5), in lieu of (5.8) we obtain

$$\begin{aligned}\tilde{\mathscr{R}}^\alpha = \rho \tilde{f}^\alpha_{.\beta}[\![\mathrm{D}_u^2 u^\beta]\!] + \{[\rho \tilde{f}^{\alpha\beta} P^{\gamma\delta} - 2P^{\alpha\beta}(t^{\gamma\delta} + t_{\mathrm{M}}^{\gamma\delta}) + t^{\alpha\beta} P^{\gamma\delta} - t^{\alpha\delta} P^{\beta\gamma}][\![\nabla_\gamma u_\delta]\!] \\ + 2[\![\mathrm{D}_u t^{\alpha\beta}]\!]\}[\![\mathrm{D}_u u_\beta]\!] - P^\alpha_{.\gamma} P^\lambda_{.\mu}[\![\mathrm{D}_u(\nabla_\lambda t^{\mu\gamma})]\!] + [\![\mathscr{G}_{\mathrm{M}}^\alpha]\!] = 0.\end{aligned} \tag{7.18}$$

The first two contributions in this equation can be evaluated on account of the expressions given in (5.11), (3.4) and (5.19). The penultimate contribution in (7.18) has already been evaluated in the purely elastic case. The last contribution, after a lengthy calculation that we shall not reproduce, gives

$$\begin{aligned}[\![\mathscr{G}_{\mathrm{M}}^\alpha]\!] = &-C_{\mathrm{M}}^{\beta\alpha\mu\sigma}[\mathscr{U}_\sigma N_\beta^\perp N_\mu^\perp + (N_\beta \phi_\mu^\Gamma + N_\mu \phi_\beta^\Gamma)\bar{U}_{\sigma|\Gamma} - b_{\Gamma\Lambda}\phi_\beta^\Gamma \phi_\mu^\Lambda \bar{U}_\sigma] \\ &-\mu \mathscr{H}^\alpha \mathscr{H}^\sigma [N_0^2 \mathscr{U}_\sigma - 2N_0 \mathscr{D}\bar{U}_\sigma - \bar{U}_\sigma \mathscr{D} N_0] \\ &+ N_0 \bar{U}_\sigma \bar{U}_\epsilon N_\beta^\perp \mathscr{R}_{\mathrm{m}}^{\beta\sigma\alpha\epsilon} + N_0^{-1}\bar{U}_\sigma \bar{U}_\epsilon N_\beta^\perp N_\mu^\perp N_\theta^\perp \mathscr{C}_{\mathrm{m}}^{\beta\alpha\sigma\mu\epsilon\theta},\end{aligned} \tag{7.19}$$

where $\boldsymbol{C}_{\mathrm{M}}$ is none other than the spatial tensor defined by (4.17) and we have set (at S^+)

$$\mathscr{R}_{\mathrm{m}}^{\beta\alpha\sigma\epsilon} = \mu(\mathscr{H}^\alpha \mathscr{H}^\sigma P^{\epsilon\beta} - \mathscr{H}^\alpha \mathscr{H}^\beta P^{\epsilon\sigma} + \mathscr{H}^\sigma \mathscr{H}^\epsilon P^{\alpha\beta} - \rho f_{\mathrm{M}}^{\beta\sigma} P^{\epsilon\alpha}) \tag{7.20}$$

and
$$\mathscr{C}_{\mathrm{m}}^{\beta\alpha\sigma\mu\epsilon\theta} = \mu(\mathscr{H}^\sigma \mathscr{H}^\alpha P^{\epsilon\beta} P^{\epsilon\theta} - \mathscr{H}^\alpha \mathscr{H}^\mu P^{\epsilon\sigma} P^{\beta\theta} + P^{\alpha\beta} \mathscr{H}^\sigma \mathscr{H}^\epsilon P^{\mu\beta} - \rho f_{\mathrm{M}}^{\mu\sigma} P^{\epsilon\alpha} P^{\beta\theta}). \tag{7.21}$$

On carrying the previously obtained partial results into (7.18), on account of the wave equation (4.13) we get

$$\begin{aligned}(1+N_0^2)\hat{H}^{\alpha\beta}\mathscr{U}_\beta - 2\rho f^{\alpha\beta} N_0 \mathscr{D}\bar{U}_\beta - (\rho f^{\alpha\sigma}\mathscr{D} N_0 - \hat{C}^{\beta\alpha\sigma\mu} b_{\Lambda\Gamma}\phi_\beta^\Gamma \phi_\mu^\Lambda)\bar{U}_\sigma - \hat{C}^{\beta\alpha\sigma\mu}(N_\beta \phi_\mu^\Gamma + N_\mu \phi_\beta^\Gamma)\bar{U}_{\sigma|\Gamma} \\ + N_0^{-1}\bar{U}_\sigma \bar{U}_\epsilon N_\beta^\perp \{N_\mu^\perp N_\theta^\perp (\tilde{\mathscr{C}}^{\beta\alpha\sigma\mu\epsilon\theta} + \mathscr{C}_{\mathrm{m}}^{\beta\alpha\sigma\mu\epsilon\theta} + 2\bar{C}^{\beta\alpha\sigma\mu}P^{\epsilon\theta}) \\ + N_0^2[\mathscr{R}_{\mathrm{m}}^{\beta\sigma\alpha\epsilon} - 2\bar{C}^{\epsilon\alpha\sigma\beta} - \rho \tilde{f}^{\alpha\sigma} P^{\epsilon\beta} + 2P^{\alpha\sigma}(t^{\epsilon\beta} + t_{\mathrm{M}}^{\epsilon\beta}) - t^{\alpha\sigma}P^{\epsilon\beta} + t^{\epsilon\alpha}P^{\sigma\beta}]\} = 0,\end{aligned} \tag{7.22}$$

where we have used the definitions (4.15) and (5.23), and set

$$\hat{C}^{\beta\alpha\sigma\mu} = \bar{C}^{\beta\alpha\sigma\mu} + C_{\mathrm{M}}^{\beta\alpha\sigma\mu} \tag{7.23}$$

so that, indeed,
$$\hat{Q}^{\alpha\sigma} = \hat{C}^{\alpha\beta\sigma\mu}\lambda_\beta \lambda_\mu. \tag{7.24}$$

Equation (7.22) can be transformed further as follows. First we notice that

$$-\rho \tilde{f}^{\alpha\sigma} N_0^2 + \hat{C}^{\alpha\mu\sigma\theta} N_\mu^\perp N_\theta^\perp = -(1+N_0^2)\hat{H}^{\alpha\sigma} - \mu \mathscr{H}^\alpha \mathscr{H}^\sigma N_0^2, \tag{7.25}$$

so that the contribution quadratic in $\bar{U}$ in (7.22) can be rewritten as

$$N_0^{-1}\bar{U}_\sigma \bar{U}_\epsilon N_\beta^\perp (N_\mu^\perp N_\theta^\perp \hat{\mathscr{C}}^{\beta\alpha\sigma\mu\epsilon\theta} + N_0^2 \hat{\mathscr{R}}^{\beta\sigma\alpha\epsilon}), \tag{7.26}$$

on account of the wave equation (4.13) if we set

$$\hat{\mathscr{C}}^{\beta\alpha\sigma\mu\epsilon\theta} = \bar{\mathscr{C}}^{\beta\alpha\sigma\mu\epsilon\theta} + \mathscr{C}_{\mathrm{M}}^{\beta\alpha\sigma\mu\epsilon\theta}, \tag{7.27}$$

$$\hat{\mathscr{R}}^{\beta\sigma\alpha\epsilon} = \bar{\mathscr{R}}^{\beta\sigma\alpha\epsilon} + \mathscr{R}_{\mathrm{M}}^{\beta\sigma\alpha\epsilon}, \tag{7.28}$$

with

$$\bar{\mathscr{C}}^{\beta\alpha\sigma\mu\epsilon\theta} = \tilde{\mathscr{C}}^{\beta\alpha\sigma\mu\epsilon\theta} + 2\bar{C}^{\beta\alpha\sigma\mu}P^{\epsilon\theta} - \bar{C}^{\alpha\mu\sigma\theta}P^{\epsilon\beta}, \tag{7.29}$$

$$\mathscr{C}_{\mathrm{M}}^{\beta\alpha\sigma\mu\epsilon\theta} = \mathscr{C}_{\mathrm{m}}^{\beta\alpha\sigma\mu\epsilon\theta} - C_{\mathrm{M}}^{\alpha\mu\sigma\theta}P^{\epsilon\beta}, \tag{7.30}$$

$$\bar{\mathscr{R}}^{\beta\sigma\alpha\epsilon} = t^{\alpha\epsilon}P^{\sigma\beta} + 2P^{\alpha\sigma}t^{\beta\epsilon} - t^{\alpha\sigma}P^{\beta\epsilon} - 2\bar{C}^{\epsilon\alpha\sigma\beta}, \tag{7.31}$$

and

$$\begin{aligned}\mathscr{R}_{\mathrm{M}}^{\beta\sigma\alpha\epsilon} &= \mathscr{R}_{\mathrm{m}}^{\beta\sigma\alpha\epsilon} - \mu\mathscr{H}^{\alpha}\mathscr{H}^{\sigma}P^{\beta\epsilon} + 2P^{\alpha\sigma}t_{\mathrm{M}}^{\beta\epsilon} \\ &= \mu[(2/\mu)\,P^{\alpha\sigma}t_{\mathrm{M}}^{\beta\epsilon} - (\rho/\mu)\,f_{\mathrm{M}}^{\beta\sigma}\,P^{\alpha\epsilon} - \mathscr{H}^{\alpha}\mathscr{H}^{\beta}P^{\sigma\epsilon} + \mathscr{H}^{\sigma}\mathscr{H}^{\epsilon}P^{\alpha\beta}].\end{aligned} \tag{7.32}$$

Then equation (7.22) transforms in the same form as (5.31), i.e.

$$\begin{aligned}2\mathscr{U}^{2}(\rho f^{\alpha\beta})_{(S^+)}\mathscr{D}\bar{U}_{\beta} + \mathscr{P}_{(1)}^{\alpha\sigma}(\mathscr{U}^{2};\,S^+,G_{2}^{W})\,\bar{U}_{\sigma} + \hat{\mathscr{P}}_{(2)}^{\alpha\sigma\epsilon}(\mathscr{U}^{2};\,S^+)\,\bar{U}_{\sigma}\bar{U}_{\epsilon} + \hat{\mathscr{P}}_{(W)}^{\alpha\sigma\Gamma}(\mathscr{U}^{2};\,S^+)\,\bar{U}_{\sigma|\Gamma} \\ = \mathscr{U}(1-\mathscr{U}^{2})^{-\frac{1}{2}}\,\hat{H}^{\alpha\sigma}\mathscr{U}_{\sigma},\end{aligned} \tag{7.33}$$

but with tensorial coefficients given by

$$\hat{\mathscr{P}}_{(1)}^{\alpha\sigma} = \mathscr{U}(1-\mathscr{U}^{2})^{\frac{1}{2}}\,(\rho\hat{f}^{\alpha\sigma}\mathscr{D}N_{0} - \hat{C}^{\beta\alpha\mu\sigma}\phi_{\beta}^{\Delta}\,\phi_{\mu}^{\Gamma}\,b_{\Delta\Gamma}), \tag{7.34}$$

$$\hat{\mathscr{P}}_{(2)}^{\alpha\sigma\epsilon} = -(1-\mathscr{U}^{2})^{\frac{1}{2}}\,(\hat{\mathscr{C}}^{\beta\alpha\sigma\mu\epsilon\theta}\lambda_{\beta}\,\lambda_{\mu}\,\lambda_{\theta} + \mathscr{U}^{2}\hat{\mathscr{R}}^{\beta\sigma\alpha\epsilon}\lambda_{\beta}), \tag{7.35}$$

and

$$\hat{\mathscr{P}}_{(W)}^{\alpha\sigma\Gamma} = \mathscr{U}\hat{C}^{\beta\alpha\mu\sigma}(\lambda_{\beta}\,\phi_{\mu}^{\Gamma} + \lambda_{\mu}\,\phi_{\beta}^{\Gamma}). \tag{7.36}$$

The alteration brought in by the magnetic field, as compared with the purely elastic case of § 5, materializes in the fact that all tensors $\boldsymbol{f}$, $\boldsymbol{C}$, $\mathscr{C}$ and $\mathscr{R}$ have a magnetic contribution as is clear from (4.15), (7.23), (7.27) and (7.28). We shall study the influence of the initial magnetic field in § 8 for a special case. For the time being, assuming as in § 5 that the wave front W is flat and the state ahead of W is spatially homogeneous, instead of (7.33) we have (cf. (5.39))

$$2\rho\mathscr{U}^{2}(\bar{U}_{\alpha}\,\hat{f}^{\alpha\beta}\mathscr{D}\bar{U}_{\beta}) + \bar{U}_{\alpha}\,\hat{\mathscr{P}}_{(2)}^{\alpha\sigma\epsilon}\bar{U}_{\sigma}\,\bar{U}_{\epsilon} = 0, \tag{7.37}$$

so that for a wave front whose elastic-disturbance polarization is along the direction of unit spatial vector m^{α}, and $\sigma = \mathscr{U}_{\mathrm{m}}\,\tau_{\mathrm{R}}$ being the distance along the spatial normal to W, we have

$$d\bar{U}_{\mathrm{m}}/d\sigma - b_{\mathrm{m}}\,\bar{U}_{\mathrm{m}}^{2} = 0 \tag{7.38}$$

with

$$b_{\mathrm{m}} = -m_{\alpha}\,\hat{\mathscr{P}}_{(2)}^{\alpha\sigma\epsilon}m_{\sigma}\,m_{\epsilon}/2\rho^{+}\hat{f}_{\mathrm{m}}\,\mathscr{U}_{\mathrm{m}}^{3} \quad (\hat{f}_{\mathrm{m}} = m_{\alpha}\,\hat{f}^{\alpha\beta}m_{\beta}), \tag{7.39}$$

and $\mathscr{U}_{\mathrm{m}}$ the positive wave speed provided by (4.34).

8. The formation of magnetoelastic shocks

Given the complexity of the problem we consider separately two special cases.

(a) *Elastically longitudinal wave fronts*

In this case $m^{\alpha} = \lambda^{\alpha}$, the invariant speed of the wave front is given by (4.36), while the corresponding magnetic-field discontinuity is given by (4.37) irrespective of the setting of the initial magnetic field. By the same token we have

$$\hat{f}_{\mathrm{m}} \rightarrow \hat{f}_{\parallel} = \lambda_{\alpha}\,\hat{f}^{\alpha\beta}\lambda_{\beta} = (\bar{f}[1+\mathscr{A}_{(\perp,\,\parallel)}^{2}])_{(S^+)}. \tag{8.1}$$

We also obtain

$$b_{\parallel} = -\hat{\mathscr{P}}_{(2)\parallel}/2\rho^{+}\hat{f}_{\parallel}\,\mathscr{U}_{\parallel}^{3} \tag{8.2}$$

and
$$\hat{\mathscr{P}}_{(2)\parallel} = -(1-\mathscr{U}_\parallel^2)^{\frac{1}{2}}\hat{\mathscr{C}}_\parallel[1+\mathscr{U}_\parallel^2(\hat{\mathscr{R}}_\parallel/\hat{\mathscr{C}}_\parallel)], \tag{8.3}$$
while on account of (7.27)–(7.32)
$$\hat{\mathscr{C}}_\parallel = \bar{\mathscr{C}}_\parallel + (\mathscr{C}_\mathrm{M})_\parallel, \quad \hat{\mathscr{R}}_\parallel = \bar{\mathscr{R}}_\parallel + (\mathscr{R}_\mathrm{M})_\parallel, \tag{8.4}$$
where the subscript $\parallel$ indicates the full projection in the $\boldsymbol{\lambda}$-direction. It is found that
$$\bar{\mathscr{C}}_\parallel = (\mathscr{C}_\mathrm{E})_\parallel + 8(\boldsymbol{C}_\mathrm{E})_\parallel - 9t_\parallel, \quad (\mathscr{C}_\mathrm{M})_\parallel = \mu(\mathscr{H}_\parallel^2 - 2\mathscr{H}_\perp^2), \tag{8.5}$$
and
$$\bar{\mathscr{R}}_\parallel = t_\parallel - 2\bar{C}_\parallel, \quad (\mathscr{R}_\mathrm{M})_\parallel = -\mu(2\mathscr{H}^2 - 3\mathscr{H}_\parallel^2) = (\mathscr{C}_\mathrm{M})_\parallel, \tag{8.6}$$
on account of (7.30), (7.21) and (4.11). Thus
$$\hat{\mathscr{C}}_\parallel = (\mathscr{C}_\mathrm{E})_\parallel + 8(\boldsymbol{C}_\mathrm{E})_\parallel - 9t_\parallel + \mu(\mathscr{H}_\parallel^2 - 2\mathscr{H}_\perp^2) \tag{8.7}$$
and
$$\hat{\mathscr{R}}_\parallel = 8t_\parallel - 2(\boldsymbol{C}_\mathrm{E})_\parallel + \mu(\mathscr{H}_\parallel^2 - 2\mathscr{H}_\perp^2). \tag{8.8}$$
We consider the special case where (6.10), (6.12) and (6.13) hold true. The model thus envisaged is the simplest one that allows us to show the effects due to an initial state of high hydrostatic pressure p_0 and those due to an arbitrarily oriented initial magnetic field. It also is the closest approximation to the classical Hookean elasticity of isotropic linear elastic bodies. We immediately have
$$\hat{\mathscr{C}}_\parallel = 8(\hat{\lambda}+2\hat{\mu}) + 9p_0 + \mu(\mathscr{H}_\parallel^2 - 2\mathscr{H}_\perp^2) \tag{8.9}$$
and
$$\hat{\mathscr{R}}_\parallel = -[8p_0 + 2(\hat{\lambda}+2\hat{\mu}) + \mu(2\mathscr{H}_\perp^2 - \mathscr{H}_\parallel^2)]. \tag{8.10}$$
We shall use the definitions (6.14) as well as the non-dimensional parameters
$$\epsilon_\mathrm{HL} = (\mu\mathscr{H}_\parallel^2/\rho c_\mathrm{L}^2)^+, \quad \epsilon_\mathrm{HT} = (\mu\mathscr{H}_\perp^2/\rho c_\mathrm{L}^2)^+, \tag{8.11}$$
which measure, respectively, the influence of the longitudinal and transverse components of the magnetic field. Those parameters are *not* supposed to be infinitesimally small. Then (4.36) yields
$$\mathscr{U}_\parallel^2 = c_\mathrm{L}^2(1+3\epsilon_p+\epsilon_\mathrm{HT})f_0^{-1}, \quad f_0 = 1+e/c^2+\beta_\mathrm{L}^2(\epsilon_p+\epsilon_\mathrm{HT}), \tag{8.12\,a, b}$$
where all speeds have been redimensionalized ($c \neq 1$) and we have set $\beta_\mathrm{L} = c_\mathrm{L}/c$. In the first of equations (8.12) there are classical (i.e. non-relativistic) contributions of p_0 and $\mathscr{H}(S^+)$ to the numerator and relativistic contributions via f_0 to the denominator. By the same token, on account of (6.14) and (8.11) we have
$$(\hat{\mathscr{C}}_\parallel/\rho)^+ = 8c_\mathrm{L}^2[1+\tfrac{9}{8}\epsilon_p+\tfrac{1}{8}(\epsilon_\mathrm{HL}-2\epsilon_\mathrm{HT})] > 0, \tag{8.13}$$
and
$$(\hat{\mathscr{R}}_\parallel/\hat{\mathscr{C}}_\parallel)^+ = -\frac{1}{4}\frac{1+4\epsilon_p-\frac{1}{2}(\epsilon_\mathrm{HL}-2\epsilon_\mathrm{HT})}{1+\frac{9}{8}\epsilon_p+\frac{1}{8}(\epsilon_\mathrm{HL}-2\epsilon_\mathrm{HT})} < 0. \tag{8.14}$$
Consequently, after rearranging some terms and setting
$$b^* = \frac{4}{c_\mathrm{L}}\frac{1+\frac{9}{8}\epsilon_p+\frac{1}{8}(\epsilon_\mathrm{HL}-2\epsilon_\mathrm{HT})}{(1+3\epsilon_p+\epsilon_\mathrm{HT})^{\frac{3}{2}}}, \tag{8.15}$$
we obtain
$$b_\parallel = b^*[f_0-\beta_\mathrm{L}^2(1+3\epsilon_p+\epsilon_\mathrm{HT})]^{\frac{1}{2}}\left\{1-\frac{\beta_\mathrm{L}^2}{4f_0}\frac{(1+3\epsilon_p+\epsilon_\mathrm{HT})[1+4\epsilon_p-\frac{1}{2}(\epsilon_\mathrm{HL}-2\epsilon_\mathrm{HT})]}{1+\frac{9}{8}\epsilon_p+\frac{1}{8}(\epsilon_\mathrm{HL}-2\epsilon_\mathrm{HT})}\right\}, \tag{8.16}$$
where b^* is the non-relativistic value of $b_\parallel$, f_0 is given by the second of equations (8.12) and is such that $f_0 = 1+O(c^{-2})$, and the contributions in β_L^2 are purely relativistic. Both the initial pressure and magnetic field alter both the non-relativistic value b^* and the relativistic corrections.

Obviously, as $|\mathscr{H}(S^+)|$ goes to zero, (8.16) reduces to (6.22) of the purely elastic case. The discussion of the behaviour of the solution of (7.38) is obviously the same as in the purely elastic case except that $\bar{\mathscr{C}}_\parallel$ is replaced by $\hat{\mathscr{C}}_\parallel$, and a strong discontinuity solution of the shock type implies that $[\![\mathscr{H}^\alpha]\!] \neq 0$ across it. However, compressive wave fronts will eventually form shocks if and only if, (6.10) and (6.13) being kept unchanged, the restrictive hypothesis materialized in the first of (6.12) is relaxed in the following conditions

$$(\mathscr{C}_{\mathrm{E}})_\parallel < 0, |(\mathscr{C}_{\mathrm{E}})_\parallel| > 8\rho^+ c_{\mathrm{L}}^2[1+\tfrac{9}{8}\epsilon_p + \tfrac{1}{8}(\epsilon_{\mathrm{HL}} - 2\epsilon_{\mathrm{HT}})]. \tag{8.17}$$

If the initial magnetic field is purely longitudinal ($\epsilon_{\mathrm{HT}} = 0$), by comparing (8.17) with (6.24*b*), we see that for a compressive wave to be generated at all magnetoelastic shocks the elastic behaviour of the material must be *more* nonlinear (that is, with a more concave stress–strain curve) in the presence of an initial magnetic field than when this magnetic field is switched off. Concomitantly, if the initial magnetic field is purely transverse then this concavity need not be as pronounced as in the purely elastic case since, then, the lower bound given by (8.17) is diminished by $2\rho^+ c_{\mathrm{L}}^2 \epsilon_{\mathrm{HT}}$ as compared with the purely elastic case. In that very sense we may therefore say that a transverse magnetic field favours the steepening of infinitesimal magnetoelastic wave fronts and the subsequent formation of magnetoelastic shocks, while a longitudinal initial magnetic field does not. The effect thus described is classical in the sense that the behaviour just observed, which consists of a balance of nonlinear elastic and magnetic effects, essentially depends on b^* or $\hat{\mathscr{C}}_\parallel$. With the nonlinearity and concavity hypothesis (8.17) valid, and setting

$$\epsilon_{\mathrm{NL}} = |(\mathscr{C}_{\mathrm{E}})_\parallel|/8\rho^+ c_{\mathrm{L}}^2 > 0, \tag{8.18}$$

as the non-dimensional positive parameter measuring the nonlinear elastic effect, we have

$$(\hat{\mathscr{C}}_\parallel/\rho)^+ = -8c_{\mathrm{L}}^2\{\epsilon_{\mathrm{NL}} - [1+\tfrac{9}{8}\epsilon_p + \tfrac{1}{8}(\epsilon_{\mathrm{HL}} - 2\epsilon_{\mathrm{HT}})]\} < 0 \tag{8.19}$$

and

$$(\hat{\mathscr{R}}_\parallel/\hat{\mathscr{C}}_\parallel)^+ = \frac{1}{4}\frac{1+4\epsilon_p - \frac{1}{2}(\epsilon_{\mathrm{HL}} - 2\epsilon_{\mathrm{HT}})}{\epsilon_{\mathrm{NL}} - [1+\frac{9}{8}\epsilon_p + \frac{1}{8}(\epsilon_{\mathrm{HL}} - 2\epsilon_{\mathrm{HT}})]} > 0. \tag{8.20}$$

The critical value of σ at which the infinitesimal compressive-wave solution breaks down (or blows up) is given by (cf. (8.16))

$$\sigma_0 = \tilde{\sigma}_0(c_{\mathrm{L}}, |\bar{U}_0|, \epsilon_p, \epsilon_{\mathrm{HL}}, \epsilon_{\mathrm{HT}}, \epsilon_{\mathrm{NL}}) < 0,$$

such that

$$\tilde{\sigma}_0 = -\frac{c_{\mathrm{L}} f_0}{4|\bar{U}_0|}\frac{(1+3\epsilon_p+\epsilon_{\mathrm{HT}})^{\frac{3}{2}}}{\epsilon_{\mathrm{NL}} - [1+\frac{9}{8}\epsilon_p + \frac{1}{8}(\epsilon_{\mathrm{HL}} - 2\epsilon_{\mathrm{HT}})]}$$
$$\times [f_0 - \beta_{\mathrm{L}}^2(1+3\epsilon_p+\epsilon_{\mathrm{HT}})]^{-\frac{1}{2}}\left\{f_0 + \tfrac{1}{4}\beta_{\mathrm{L}}^2\frac{(1+3\epsilon_p+\epsilon_{\mathrm{HT}})\,[1+4\epsilon_p - \frac{1}{2}(\epsilon_{\mathrm{HL}} - 2\epsilon_{\mathrm{HT}})]}{\epsilon_{\mathrm{NL}} - (1+\frac{9}{8}\epsilon_p + \frac{1}{8}(\epsilon_{\mathrm{HL}} - 2\epsilon_{\mathrm{HT}})]}\right\}^{-1}. \tag{8.21}$$

The relativistic effects related to both p_0 and $\mathscr{H}(S^+)$ appear in this formula in an intricate manner in the factor f_0 in the numerator and in the three terms at negative powers. Note that, on account of (6.14) and (8.11)

$$f_0 - \beta_{\mathrm{L}}^2(1+3\epsilon_p+\epsilon_{\mathrm{HT}}) = 1+\epsilon/c^2 - \beta_{\mathrm{L}}^2(1+2\epsilon_p). \tag{8.22}$$

A similar transformation holds good for the last term in (8.21). Finally, for the most interesting case of a purely transverse initial magnetic field we obtain

$$\tilde{\sigma}_0 = -\frac{c_{\mathrm{L}} f_0}{4|\bar{U}_0|}\frac{(1+3\epsilon_p+\epsilon_{\mathrm{HT}})^{\frac{3}{2}}[1+\epsilon/c^2 - \beta_{\mathrm{L}}^2(1+2\epsilon_p)]^{-\frac{1}{2}}}{[\epsilon_{\mathrm{NL}} - (1+\frac{9}{8}\epsilon_p - \frac{1}{4}\epsilon_{\mathrm{HT}})]\,[1+\epsilon/c^2 + \beta_{\mathrm{L}}^2\mathscr{F}(\epsilon_p, \epsilon_{\mathrm{HT}}, \epsilon_{\mathrm{NL}})]}, \tag{8.23}$$

with f_0 given by (8.12*b*) and

$$\mathscr{F}(\epsilon_p, \epsilon_{\mathrm{HT}}, \epsilon_{\mathrm{NL}}) = \epsilon_p + \epsilon_{\mathrm{HT}} + \frac{(1+3\epsilon_p+\epsilon_{\mathrm{HT}})\,(1+4\epsilon_p+\epsilon_{\mathrm{HT}})}{4[\epsilon_{\mathrm{NL}} - (1+\frac{9}{8}\epsilon_p) - \frac{1}{4}\epsilon_{\mathrm{HT}}]}. \tag{8.24}$$

This completes the solution for the growth of elastically longitudinal magnetoelastic wave fronts in a nonlinear elastic body in the presence of an initial state of high hydrostatic pressure and a transverse magnetic field.

(*b*) *Elastically transverse wave fronts*

In this case $m^\alpha = d^\alpha$, $d_\alpha P^{\alpha\beta}\lambda_\beta = 0$. The invariant speed of the wave front is given by (4.38) whereas the magnetic-field discontinuity is given by (4.39) irrespective of the setting of the initial magnetic field. We have

$$b_\perp = -\hat{\mathscr{P}}_d/2\rho^+\hat{f}_d\,\mathscr{U}_\perp^3, \tag{8.25}$$

with

$$\hat{\mathscr{P}}_d = d_\alpha \hat{\mathscr{P}}_{(2)}^{\alpha\sigma\epsilon} d_\sigma d_\epsilon, \quad \hat{f}_d = d_\alpha \hat{f}^{\alpha\beta} d_\beta. \tag{8.26}$$

Assuming that the working hypotheses (6.10), (6.12) and (6.13) hold true, after some calculation we obtain ($\beta_{\mathrm{T}} = c_{\mathrm{T}}/c$)

$$\hat{f}_d = 1+\epsilon/c^2+\zeta_{\mathrm{LT}}\beta_{\mathrm{T}}^2[\epsilon_p+\epsilon_{\mathrm{HL}}+\epsilon_{\mathrm{HT}}(1-\cos^2\theta)], \tag{8.27}$$

$$\mathscr{U}_\perp^2 = c_{\mathrm{T}}^2[1+\zeta_{\mathrm{LT}}(\epsilon_p+\epsilon_{\mathrm{HL}})]\hat{f}_d^{-1}, \tag{8.28}$$

$$\bar{\mathscr{C}}_d = d_\alpha d_\sigma d_\epsilon \bar{\mathscr{C}}^{\beta\alpha\sigma\mu\epsilon\theta}\lambda_\beta\lambda_\mu\lambda_\theta \equiv 0, \tag{8.29}$$

$$\bar{\mathscr{R}}_d = -2\bar{C}^{\epsilon\alpha\sigma\beta} d_\alpha d_\sigma d_\epsilon d_\beta \equiv 0, \tag{8.30}$$

$$(\mathscr{C}_{\mathrm{M}})_d = d_\alpha d_\sigma d_\epsilon \mathscr{C}_{\mathrm{M}}^{\beta\alpha\sigma\mu\epsilon\theta}\lambda_\beta\lambda_\mu\lambda_\theta \equiv 0, \tag{8.31}$$

and

$$(\mathscr{R}_{\mathrm{M}})_d = d_\alpha d_\sigma d_\epsilon \mathscr{R}_{\mathrm{M}}^{\beta\sigma\alpha\epsilon}\lambda_\beta$$
$$= 2t_{\mathrm{M}}^{\beta\epsilon} d_\epsilon\lambda_\beta - \rho f_{\mathrm{M}}^{\beta\sigma} d_\sigma\lambda_\beta - (\mathscr{H}_\perp\cdot\boldsymbol{d})\,\mathscr{H}_\parallel = 2\mu(\mathscr{H}_\perp\cdot\boldsymbol{d})\,\mathscr{H}_\parallel. \tag{8.32}$$

Since both $\bar{\mathscr{C}}_d$ and $(\mathscr{C}_{\mathrm{M}})_d$ are nil, $b_\perp \equiv 0$ in this case and elastically transverse wave fronts travel with constant amplitude throughout S^+ in spite of the presence of an initial magnetic field.

Consider now that $(\mathscr{C}_{\mathrm{E}})_d = d_\alpha d_\sigma d_\epsilon \mathscr{C}_{\mathrm{E}}^{\beta\alpha\sigma\mu\epsilon\theta}\lambda_\beta\lambda_\mu\lambda_\theta \neq 0$ *a priori*, hence that the body behaves *nonlinearly*, while (6.10) and (6.13) are kept unchanged. We have then

$$b_\perp = \frac{(1-\mathscr{U}^2)^{\frac{1}{2}}}{2\hat{f}_d\,\mathscr{U}_\perp^3}\frac{(\mathscr{C}_{\mathrm{E}})_d}{\rho^+}\left\{1+\mathscr{U}_\perp^2\frac{2\mu\,|\mathscr{H}_\perp|\,\mathscr{H}_\parallel\cos\theta}{(\mathscr{C}_{\mathrm{E}})_d}\right\} \tag{8.33}$$

in non-dimensional speeds. Setting

$$\hat{\epsilon}_{\mathrm{NL}} = (\mathscr{C}_{\mathrm{E}})_d/2\rho^+c_{\mathrm{L}}^2 \tag{8.34}$$

we obtain with dimensionalized speeds

$$b_\perp = b^{**}\{\hat{f}_d-\beta_{\mathrm{T}}^2[1+\zeta_{\mathrm{LT}}(\epsilon_p+\epsilon_{\mathrm{HL}})]\}^{\frac{1}{2}}$$
$$\times\{1+(\beta_{\mathrm{T}}^2/\hat{f}_d\hat{\epsilon}_{\mathrm{NL}})\,[1+\zeta_{\mathrm{LT}}(\epsilon_p+\epsilon_{\mathrm{HL}})]\,(\epsilon_{\mathrm{HL}}\,\epsilon_{\mathrm{HT}})^{\frac{1}{2}}\cos\theta\}, \tag{8.35}$$

where

$$b^{**} = \frac{\hat{\epsilon}_{\mathrm{NL}}}{c_{\mathrm{L}}}\left[\frac{\zeta_{\mathrm{LT}}}{1+\zeta_{\mathrm{LT}}(\epsilon_p+\epsilon_{\mathrm{HL}})}\right]^{\frac{3}{2}}, \tag{8.36}$$

is the non-relativistic value of $b_\perp$. Since the critical value σ_0 of the ray parameter σ behaves like $b_\perp^{-1}$, an initial magnetic field that possesses a non-zero longitudinal component has the effect of increasing this critical value at the non-relativistic order. That is, *a longitudinal magnetic field delays to some extent the occurrence of a shock solution.* The fully relativistic expression of σ_0 is given by

$$\sigma_0 = \frac{c_{\mathrm{L}}\hat{f}_d}{|\bar{U}_0|}\frac{[\zeta_{\mathrm{LT}}^{-1}+(\epsilon_p+\epsilon_{\mathrm{HL}})]^{\frac{3}{2}}}{\hat{\epsilon}_{\mathrm{NL}}[1+\epsilon/c^2+\zeta_{\mathrm{LT}}\beta_{\mathrm{T}}^2\epsilon_{\mathrm{HT}}(1-\cos^2\theta)]}[1+\epsilon/c^2+\beta_{\mathrm{T}}^2\mathscr{F}'(\epsilon_p,\epsilon_{\mathrm{HL}},\epsilon_{\mathrm{HT}},\hat{\epsilon}_{\mathrm{NL}})]^{-1}, \tag{8.37}$$

where $|\bar{U}_0| = |\bar{U}_d(\sigma=0)|$, $\hat{f}_d$ is given by (8.27), and we have set

$$\mathscr{F}'(\epsilon_p,\epsilon_{\mathrm{HL}},\epsilon_{\mathrm{HT}},\hat{\epsilon}_{\mathrm{NL}})$$
$$= \zeta_{\mathrm{LT}}[\epsilon_p+\epsilon_{\mathrm{HL}}+\epsilon_{\mathrm{HT}}(1-\cos^2\theta)]-\hat{\epsilon}_{\mathrm{NL}}^{-1}[1+\zeta_{\mathrm{LT}}(\epsilon_p+\epsilon_{\mathrm{HL}})](\epsilon_{\mathrm{HL}}\,\epsilon_{\mathrm{HT}})^{\frac{1}{2}}\cos\theta. \tag{8.38}$$

If the initial magnetic field is purely longitudinal this reduces to

$$\sigma_0 = \frac{c_L}{|\bar{U}_0|} \frac{(\zeta_{LT}^{-1} + \epsilon_p + \epsilon_{HL})^{\frac{3}{2}}}{\hat{\epsilon}_{NL}(1 + \epsilon/c^2)^{\frac{1}{2}}}, \tag{8.39}$$

whereas for a purely transverse initial magnetic field, we have

$$\sigma_0 = \frac{c_L}{|\bar{U}_0|} \frac{(\zeta_{LT}^{-1} + \epsilon_p)^{\frac{3}{2}}}{\hat{\epsilon}_{NL}[1 + \epsilon/c^2 + \zeta_{LT}\beta_T^2 \epsilon_{HT}(1 - \cos^2\theta)]^{\frac{1}{2}}}. \tag{8.40}$$

If in addition the transverse elastic disturbance is polarized in such a way that $\bar{U}^\alpha$ is parallel or antiparallel to $\mathscr{H}^\alpha(S^+) \equiv \mathscr{H}_\perp^\alpha(S^+)$, then the initial magnetic field has no effect on σ_0 which then depends only on ϵ_p and $\hat{\epsilon}_{NL}$.

To conclude this section, in view of (8.36), (8.39) and (8.40), we can say that while a longitudinal magnetic field may delay to some extent the formation of transverse magnetoelastic shocks in a nonlinear elastic body in a state of high hydrostatic pressure at the *non-relativistic order*, a purely transverse magnetic field, if not aligned with the transverse elastic disturbance, may favour the formation of such shocks, *but at the relativistic order*, since it clearly decreases the value of the critical parameter σ_0 (cf. (8.40)). In any event, a nonlinear elastic behaviour is needed for those effects to show up. Otherwise, an infinitesimal, elastic, transverse wave front traverses the initial state without change in its amplitude.

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